LECTURE 34

Positive self-adjoint operators.

"positive-semidefinite" in linear algebra

Def A selfadjoint op. T on a Hilbert space T is called positive if its quadratic form is nonnegative:
$$\langle Tx, x\rangle \geq 0 \quad \text{for all } x \in H.$$

Examples: 1) $T^2 \geq 0$ $\forall$ selfadj T $(\langle T^2x, x\rangle = \langle Tx, Tx\rangle \geq 0)$; 2) Compact self-adj. T with eigenvalues ≥ 0 (by Spectral Thm)

Def (Order) We will say $A \leq B$ if $B - A \geq 0$

This defines a partial order on $\mathcal{L}(H,H)$

~~Example: $A^2 \geq 0$ $\forall A$ selfadj $(\langle A^2x, x\rangle = \langle Ax, Ax\rangle \geq 0)$.~~

~~Properties:~~

Lemma $\|A\| \leq M$ iff $-MI \leq A \leq MI$

Proof $A \leq MI \Leftrightarrow MI - A \geq 0 \Leftrightarrow \langle (MI - A)x, x\rangle \geq 0 \;\forall x \Leftrightarrow M\langle x, x\rangle - \langle Ax, x\rangle \geq 0$

$\Leftrightarrow \langle Ax, x\rangle \leq M\|x\|$.

Similarly, $-MI \leq A \Leftrightarrow \langle Ax, x\rangle \geq -M\|x\|$.

~~Hence This completes the proof since~~

Hence $-MI \leq A \leq MI \Leftrightarrow |\langle Ax, x\rangle| \leq M\|x\| \;\forall x \Leftrightarrow$

$\|A\| \leq M$ by Thm p. 131. QED

- We proved actually that $mI \leq A \leq MI \Leftrightarrow m \leq \langle Ax, x\rangle \leq M$ for all $x \in S_H$. (p. 132)

A reformulation of the Thm on the spectrum of self-adjoint operators then reads as:

THM Let A be a selfadjoint op. on H. Let m, M be the smallest/largest numbers s.t. $mI \leq A \leq MI$.

Then 1) $\sigma(A) \subseteq [m, M]$.

2) $m, M \in \sigma(A)$

Cor A selfadj; $A \geq 0$ iff $\sigma(A) \subseteq [0, \infty)$.

Polynomials of operators

Here $T \in L(X,X)$ an arbitrary op. in a Banach space X.

Def For a polynomial $p(t) = \cancel{a_0 + a_1 t + \dots +}\ a_0 + a_1 t + \dots + a_n t^n$,
we define the ~~polynomial of operator~~ $p(T) = a_0 I + a_1 T + \dots + a_n T^n \quad \in L(X,X)$.

Properties: $\forall$ polynomials f, g,

$$(f+g)(T) = f(T) + g(T);$$

$$(fg)(T) = f(T)g(T);$$

In other words, for a given $T \in L(X,X)$, the mapping $f \mapsto f(T)$ is a homomorphism

$$P(t) \longrightarrow L(X,X).$$

Lemma For every polynomial p, the operator $p(T)$ is invertible if and only if $p(t) \neq 0$ for all $t \in \sigma(T)$.

Proof Factorize $p(t) = a_n (t - t_1) \cdots (t - t_n)$, where t_i are the roots of $p(t)$.
Then $\quad p(T) = a_n (T - t_1 I) \cdots (T - t_n I)$.
~~The~~ Invertibility of $p(T)$ is equivalent to invertibility of each factor $T - t_i I$ (check!)
which in turn is equivalent to $t_i \notin \sigma(T)$ for all i.

QED.

Spectral Mapping Thm. For every polynomial $p(t)$ and operator $T \in L(X,X)$,

$$\sigma(p(T)) = p(\sigma(T)).$$

$= \{p(t) : t \in \sigma(T)\}$.

Proof $\lambda \in \sigma(p(T)) \iff p(T) - \lambda I$ is invertible

$= (p-\lambda)T$

$\overset{\text{Lemma}}{\iff} (p-\lambda)(t) = 0$ for some $t \in \sigma(T)$

$\Updownarrow$

$p(t) = \lambda$

$\iff \lambda \in p(\sigma(T))$.

QED

Cor For selfadjoint operators T,
$$\|p(T)\| = \sup_{t \in \sigma(T)} |p(t)|$$

(This generalizes $\|T\| = \sup_{t \in \sigma(T)} |t| = r(T)$ for self-adj. is Cor. p. 134).

Proof $\|p(T)\| = r(p(T))$

$$= \sup_{t \in \sigma(p(T))} |t| = \sup_{t \in p(\sigma(T))} |t| \quad \text{by SMT}$$

$$= \sup_{s \in \sigma(T)} |p(s)|.$$

QED.

Continuous functions of operators

- Let T ~~&~~ be a self-adjoint operator on a Hilbert space H, and f be a continuous function on $\sigma(T)$. ~~What is f~~
- We would like to define $f(T)$.
- First, one can extend $f(t)$ to a continuous function (also called f) defined on an interval ~~[a,b] con~~ $[m, M]$ containing the spectrum. (Show!)
- Then one can use Weierstrass thm:

$$\exists \text{ polynomials } p_n(t) \to f(t) \text{ uniformly on } \sigma(T)$$

~~**Lemma** ~~Assume $p_n(t)$ converges in~~~~
~~i) $p_n(T)$ converges in $L(H,H)$~~
~~ii) If $p_n(t) \to f(t)$, $q_n(t) \to f(t)$ uniformly on $\sigma(T)$ then the limit .~~

Lemma/Def
1) The sequence $p_n(T)$ converges in $L(H,H)$ to a limit called $f(T)$
2) The limit does not depend on the choice of ~~that~~ the sequence of polynomials p_n.

Proof 1) By ~~the~~ completeness of $L(H,H)$ it suffices to check that $p_n(T)$ is a Cauchy sequence. By Cor p. 143,

$$\|p_n(T) - p_m(T)\| = \|(p_n - p_m)(T)\| = \sup_{t \in \sigma(T)} |(p_n - p_m)(t)| \to 0 \quad \text{as } n, m \to \infty.$$

2) Similarly, if $q_n(t) \to f(t)$ uniformly on $\sigma(T)$ then

$$\|p_n(T) - q_n(T)\| = \sup_{t \in \sigma(T)} |(p_n - q_n)(t)| \to 0 \quad \text{as } n \to \infty.$$

QED.

Properties (By passing to limit in Properties on p.142): $\forall f,g \in C(\sigma(T))$,

$$\begin{cases}(f+g)(T)=f(T)+g(T);\\ (fg)(T)=f(T)g(T).\end{cases}$$

In other words, the mapping $f \mapsto f(T)$ is a homomorphism

$$C(\sigma(T)) \longrightarrow L(H,K)$$

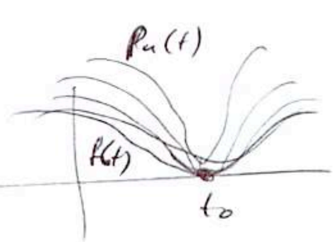
Exercise: $\|f(T)\| = \sup_{t\in\sigma(T)} |f(t)| \left(= \|f\|_\infty\right)$. (use Cor p143)

Example $\frac{1}{T}$ is the inverse T^{-1}.

Lemma (~~Analog~~ Generalization of lemma p.142): Let $T \in L(H,H)$ be a selfadj op. For every $f \in C(\sigma(T))$, the operator $f(T)$ is invertible ~~iff~~ if and only if $f(t) \neq 0$ for all $t \in \sigma(T)$.

Proof ($\Leftarrow$) Assume $f(t) \neq 0$ for $t \in \sigma(T)$.

Then $\frac{1}{f(t)} \in C(\sigma(T))$, and by Properties above, $\frac{1}{f(T)}$ is the inverse of T.



($\Rightarrow$) Assume $f(t_0)=0$ for some $t_0 \in \sigma(T)$.

Let $p_n(t) \to f(t)$ uniformly on $\sigma(T)$

WLOG $p_n(t_0)=0$ (why? — by slight translations)

By SMT for polynomials (lemma p142), the operators $p_n(T)$ are not invertible.

The set of non-invertible polynomials is closed (in $L(H,K)$) (HW 11/20)

$\Rightarrow f(T)$ is non-invertible. QED.

SPECTRAL MAPPING THM (For continuous functions)

Let T be a self-adj. in a Hilbert space H, let f be a continuous function on $\sigma(T)$. Then

$$\sigma(f(T)) = f(\sigma(T)).$$

Proof: ~~repeat the proof~~ follows from Lemma in exactly the same way as SMT for polynomials p.143. QED.