For \( u \leq f(t) \leq M \) for all \( t \in \mathbb{R} \).

Then \( u \leq f(T) \leq M \) for all self-adjoint \( T \).

Proof. By Thm. p. 141, it suffices to the conclusion would follow if we prove that

\[ 0 \leq f(T) \leq [u, M]. \]

But this follows from the S.M.T.:

\[ \sigma(f(T)) = f(\sigma(T)) \leq [u, M]. \]

Q.E.D.

Remark. The assumption can be weakened to

\[ u \leq f(t) \leq M \text{ for all } t \in \sigma(T). \]

Recall polar decomposition for complex numbers: \( z = r e^{i\theta} = |z| e^{i\theta} \).

**Square root**

Let \( T \) be a positive self-adjoint operator.

Then \( \sigma(T) \subseteq [0, \infty) \).

The function \( f(t) = \sqrt{t} \) is continuous on \([0, \infty)\).

Hence (using cor. above), \( \sqrt{T} \) is a positive self-adjoint operator, such that \( (\sqrt{T})^2 = T \).

This follows from \( f(t)g(t) = fg(t) \).

**Exercise** The square root is unique, i.e. if \( A^2 = T \) for some \( A \geq 0 \) then \( A = \sqrt{T} \) [Kap 359]

**Exercise** The product of positive commutative operators is positive:

\[ \langle ABx, x \rangle = \langle \sqrt{A}B \sqrt{A}Bx, x \rangle = \langle \sqrt{A}Bx, \sqrt{A}Bx \rangle \geq 0. \]

-146
Modulus  Let $T \in L(H, H)$ be arbitrary.

Then $T^* T$ is a positive self-adjoint on $H$,
hence it has the (unique) positive square root

$$|T| := \sqrt{T^* T}$$  \hspace{1cm} \text{("Modulus" of $T")}$$

\textbf{Examples:} 1) $T = (\alpha \omega \omega^* R) \Rightarrow |T| = (|\alpha| I \omega \omega^* R)$.

\textbf{Example:} 2) $T = (\alpha \omega \omega^* R) \Rightarrow |T| = (|\alpha| I \omega \omega^* R)$.

\textbf{Example:} 3) Right shift in $l_2 = \mathbb{R} I$.

\textbf{Lemma:} For every $x \in H$,

$$\|T^* x\| = \|T x\|.$$

\textbf{Proof:}

$$\|T x\|^2 = \langle T^* x, T^* x \rangle = \langle T T^* x, x \rangle = \langle T^* T x, x \rangle = \langle T x, T x \rangle = \|T x\|^2$$  \hspace{1cm} \text{QED}$$

\textbf{Lemma shows that the linear operator}

$$U : H \rightarrow H$$

motivates us to consider an isometry

$$U : \|T^* x\| \rightarrow \|T x\|.$$

1) $U$ is well defined on $\text{Im}(T^*)$.

(Indeed, if $\|T^* x\| = \|T^* y\|$ then $\|T^* x - y\| = 0 \Rightarrow \|T^* x\|^2 = 0$ by \textbf{Lemma}).

\text{That is, $T x = Ty$.}

2) $U$ is linear.

3) $U$ is an isometry, i.e. $\|U z\| = \|z\|$ for all $z$ (this is \textbf{Lemma}).

4) $\text{Im}(U) = \text{Im}(T)$ (obvious).

We have proved:  

---

147---
TKH (Polar decomposition)

Let $T \in L(H, H)$ be arbitrary.
Then there exists a bijective isometry $U \in L(\text{Im}(T), \text{Im}(T))$
such that
$$T = U|T|$$

Hence, such $U$ is unique.

**Proof (sketch; the uniqueness is left to prove).**

**Remarks:**
1. $U$ is unique: $T_x = U|T|x$ means that
   $$U : |T|x \mapsto T_x,$$
   hence $U$ is uniquely defined on $\text{Im}(T)$.

2. $U$ cannot be in general extended to an
   a bijective isometry on $H$.

Indeed, if $T = \text{right shift in } l^2$ then
$$|T| = I \Rightarrow U = T \text{ by polar decomposition},$$

For what operator $T$ is this possible? !

---

**Unitary operators**

**Def.** A unitary operator on a Hilbert space $H$ is a bijective isometry $U \in L(H, H)$.

**Prop.** $U$ is unitary if and only if $U^*U = I$ and $UU^* = I$.

Thus, unitary operators are similar to complex numbers.

Thus, unitary operators are similar to complex numbers $\Re z = 1, \Im z = 0$.

**Proof.** (sketch)
Unitary operators

Def. \( U \in \mathcal{L}(H, H) \) is a unitary operator if \( U \) is a bijective isometry.

Examples: 1) Unitary (orthogonal) matrices \( n \times n \)  
2) Right shift on \( l^2 \) (but not left!)  
3) Isometry between separable Hilbert spaces.

Remark. A unitary operator preserves the norm: \( \|Ux\| = \|x\| \), hence also the inner product \( \langle Ux, Uy \rangle = \langle x, y \rangle \) (by polarization identity).

Prop. \( U \in \mathcal{L}(H, H) \) is unitary if and only if

\[ U^* U = UU^* = I, \]

i.e., \( U \) is invertible and \( U^{-1} = U^* \).

Remark. This is analogous to the complex numbers: \( \bar{z} \overline{\bar{z}} = |z|^2 \), \( |z|^2 = 1 \).

Proof. 

\((\Rightarrow)\) \( \langle U^* U x, x \rangle = \langle U x, U x \rangle = \|U x\|^2 = \|x\|^2 = \langle x, x \rangle \)

Hence the quadratic forms of \( U^* U \) and \( I \) coincide.

Then \( U^* U = I \) (see Remark 1 p. 130)

Similarly for \( U U^* \).

\((\Leftarrow)\) Since \( U \) is invertible, it is bijective. To check bijectivity, note that

\[ \|U x\|^2 = \langle U x, U x \rangle = \langle U^* U x, x \rangle = \langle x, x \rangle = \|x\|^2. \]

Prop. The spectrum of a unitary operator \( U \) lies on the unit circle:

\[ \sigma(U) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}. \]

Proof. Since \( U \) is an isometry, \( \|U\| = \|U^*\| = 1 \).

Then the spectral radius \( r(U) \leq \|U\| = 1 \). Hence \( \sigma(U) \subseteq \{ |\lambda| \leq 1 \} \).

On the other hand, if \( |\lambda| < 1 \) then \( U - \lambda I \) is invertible, as \( U^{-1} (U - \lambda I) = I - \lambda U^{-1} \) is invertible by von Neumann series since \( \|U^{-1}\| = \|\lambda| < 1 \).

\( U - \lambda I \) is invertible. Q.E.D.
Prop: Eigenvectors corresponding to distinct eigenvalues of $U$ are orthogonal.

Proof: \[ \langle x, y \rangle = \langle Ux, Uy \rangle = \langle Ax, Ay \rangle = \lambda \mu \langle x, y \rangle. \]

Hence if \( \langle x, y \rangle \neq 0 \) then \( \lambda \mu = 1 \). Since \( |\lambda| = |\mu| = 1 \Rightarrow \lambda = \mu. \)

Q.E.D

Thm: For invertible operators $T \in L(H, H)$, the polar decomposition holds:

\[ T = U |T| \]

for some $U$ unitary.

Proof: $T$ is invertible $\Rightarrow T^*T$ is invertible $\Rightarrow T^*T$ is invertible

$\Rightarrow |T| = T^*T$ is invertible (inverses exist)

$\Rightarrow$ $\text{Im} T = T^*T = H.$

Q.E.D

Remark: This form of polar decomposition holds also for:

- normal operators;
- compact + scalar operators;
- generally $U^* T U = \text{dilute} T = \text{dilute} T^*$ (Kao).