Cor (Range) Assume that ~~...~~

$$m \le f(t) \le M \quad \text{for all } t.$$

Then

$$m I \le f(T) \le M I. \quad \text{~~for all self-adjoint T~~}$$

Proof. By Thm p. 141, ~~it suffices to~~ the conclusion would follow if we prove that

$$\sigma(f(T)) \subseteq [m, M].$$

But this follows from the S.M.T.:

$$\sigma(f(T)) = f(\sigma(T)) \subseteq [m, M]. \qquad \text{QED}$$

Remark clearly, The assumption can be weakened to

$$m \le f(t) \le M \quad \text{for all } t \in \sigma(T).$$

Recall polar decomp. for complex numbers: $z = e^{i \arg z} \cdot |z|$.

Polar decomposition

- Square root

Let T be a positive self-adjoint operator.

Then $\sigma(T) \subset [0, \infty)$.

The function $f(t) = \sqrt{t}$ is continuous on $[0, \infty)$.

Hence (using Cor. above),

$$\boxed{\sqrt{T} \text{ is a positive self-adjoint operator, such that } \left(\sqrt{T}\right)^2 = T.}$$

this follows from $f(T) g(T) = fg(T)$.

Exercise The positive square root is unique, i.e. if $A^2 = T$ for some $A \ge 0$ then $A = \sqrt{T}$ [Kap. 334]

Exercise The product of positive commuting operators is positive

(Ka p. 316): A, B commute $\Rightarrow \sqrt{A}, \sqrt{B}$ commute $\Rightarrow$

$$\langle ABx, x\rangle = \langle \sqrt{AB} \cdot \sqrt{AB}\, x, x\rangle = \langle \sqrt{A}\sqrt{B} \cdot \sqrt{A}\sqrt{B}\, x, x\rangle$$

$$= \langle \sqrt{A}\sqrt{B}\, x, \sqrt{A}\sqrt{B}\, x\rangle \ge 0.$$

<u>Modulus</u> Let $T \in \mathcal{L}(H,H)$ be arbitrary.

Then T^*T is a positive self-adjoint on H,

hence it has the ~~sqr~~ (unique) positive square root

$$|T| := \sqrt{T^*T} \qquad (\text{"Modulus" of } T).$$

Examples 1) Diag. $T = \begin{pmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & \\ & & \ddots \end{pmatrix} \Rightarrow |T| = \begin{pmatrix} |\lambda_1| & & 0 \\ 0 & |\lambda_2| & \\ & & \ddots \end{pmatrix}$ 2) Mult. in $L_2[0,1]$ 3) $|\text{Right shift in } \ell_2| = I$.

$$\boxed{\textbf{Lemma} \quad \text{For every } x \in H, \quad \| \, |T| x \| = \|Tx\|.}$$

<u>Proof</u> $\| \, |T|x \|^2 = \langle |T|x, |T|x \rangle = \langle |T|^2 x, x \rangle = \langle T^*Tx, x \rangle$

$= \langle Tx, Tx \rangle = \|Tx\|^2$ QED.

• ~~Lemma shows that the linear operator~~

~~U:~~

Lemma motivates us to consider an isometry

$$U : |T|x \mapsto Tx.$$

1). U is well <u>defined</u> on $\mathrm{Im}(|T|)$.

(Indeed, if $|T|x = |T|y$ then $|T|(x-y) = 0 \Rightarrow \|T(x-y)\| = 0$ by Lemma $\Rightarrow Tx = Ty$.)

2) U is a <u>linear</u> operator

3) U is an <u>isometry</u>, i.e. $\|Uz\| = \|z\|$ for all z (this is Lemma)

4) $\mathrm{Im}(U) = \mathrm{Im}(T)$ (obvious).

We have proved:

THM (Polar decomposition)

Let $T \in L(H,H)$ be arbitrary.
Then there exists a bijective isometry $U \in L(\mathrm{Im}(|T|), \mathrm{Im}(T))$ such that

$$T = U|T|.$$

~~Moreover, such U is unique.~~

~~Proof (only the uniqueness is left to prove).~~

Remarks 1) U is unique : $Tx = U|T|x$ means that
$U : |T|x \mapsto Tx$, hence U is uniquely defined on $\mathrm{Im}\,|T|$.

2) U can not be in general extended to a bijective isometry on H.
Indeed, if $T =$ right shift in l_2 then
$|T| = I \Rightarrow$ by Polar decomposition,
$U = T$ (defined on H but not isometry – image $\neq H$).

For what operators T is this possible?

Unitary operators

Def A unitary operator on a Hilbert space H is a bijective isometry $U \in L(H,H)$.

Prop U is unitary if and only if $U^*U = I$ and $UU^* = I$

(~~This is similar to complex numbers~~)

Thus unitary operators are similar to complex numbers s.t. $|z| = 1$: $\bar{z} z = 1$.

Proof • ($\Leftarrow$)

Unitary operators

Def $U \in \mathcal{L}(H,H)$ is a <u>unitary operator</u> if U is a bijective isometry.

<u>Examples</u>: 1) Unitary (orthogonal) matrices $n \times n$ — symmetries, permutations, rotations of $\mathbb{C}^n$.

2) Right shift on ℓ_2 (but not left!).

3) Isometry between ∀ separable Hilbert spaces.

<u>Remark</u> A unitary operator preserves the norm: $\|Ux\| = \|x\|$, hence also the inner product $\langle Ux, Uy \rangle = \langle x, y \rangle$, (by polarization identity)

Prop $U \in \mathcal{L}(H,H)$ is unitary if and only if
$$U^*U = UU^* = I,$$
i.e. if U is invertible and $U^{-1} = U^*$.

<u>Remark</u>: this is analogous to the unit complex numbers: $\bar{z}z = z\bar{z} = 1$, $|z| = 1$.

<u>Proof</u> ($\Rightarrow$) $\langle U^*Ux, x \rangle = \langle Ux, Ux \rangle = \|Ux\|^2 = \|x\|^2 = \langle x, x \rangle$

Hence the quadratic forms of U^*U and of I coincide.

Then $U^*U = I$ (see Remark 1 p. 130)

Similarly for UU^*.

($\Leftarrow$) Since U is invertible, it is bijective. To check isometry, note that
$$\|Ux\|^2 = \langle Ux, Ux \rangle = \langle U^*Ux, x \rangle = \langle x, x \rangle = \|x\|^2.$$
QED

Prop The spectrum of a unitary operator U lies on the unit circle:
$$\sigma(U) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

<u>Proof</u> ~~We have~~ Since U is an isometry, $\|U\| = \|U^{-1}\| = 1$.

Then the spectral radius $r(U) \le \|U\| = 1$. Hence $\sigma(U) \subseteq \{|\lambda| \le 1\}$.

On the other hand, if $|\lambda| < 1$ then $U - \lambda I$ is invertible then

$U^{-1}(U - \lambda I) = I - \lambda U^{-1}$ is invertible by Von Neumann series since $\|\lambda U^{-1}\| = |\lambda| < 1$.

$\Rightarrow U - \lambda I$ is invertible. QED.

<u>Exercise</u>: Prove the stronger Polar decomp. for invertible $T \in \mathcal{L}(H,H)$: $T = U|T|$ for some unitary U

(Show that since T is invertible, $|T|$ is invertible $\Rightarrow$ $\operatorname{Im}|T| = \operatorname{Im} T = H$)

Prop Eigenvectors corresponding to distinct eigenvalues of U are orthogonal.

Proof $\langle x,y\rangle = \langle Ux, Uy\rangle = \langle \lambda x, \mu y\rangle = \lambda\bar{\mu}\,\langle x,y\rangle$.

If $\langle x,y\rangle \neq 0$ then $\lambda\bar{\mu} = 1$. Since $|\lambda| = |\mu| = 1 \Rightarrow \lambda = \mu$.

QED

THM For invertible operators $T \in L(H,H)$, the stronger polar decomposition holds:

$$T = U|T| \quad \text{for some } U \text{ unitary.}$$

Proof T is invertible $\Rightarrow T^*$ is invertible $\Rightarrow T^*T$ is invertible

$\Rightarrow |T| = \sqrt{T^*T}$ is invertible

$\Rightarrow \operatorname{Im} T = \operatorname{Im} |T| = H$.

QED

Remark This form of polar decomposition holds also for

normal operators;

compact + scalar operators;

generally $\forall T$ s.t. $\dim\ker T = \dim\ker T^*$ (Ka)