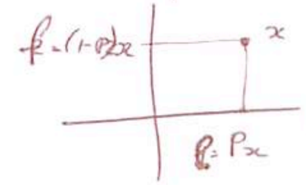
LECTURE 36

Orthogonal projections

- Let $H = E \oplus F$ be an orthogonal decomposition of a Hilbert space, i.e. $E \subseteq H$ is a closed linear subspace and $F = E^{\perp}$.



- Then every $x \in H$ can be written as

$$x = e + f; \quad e \in E, \; f \in F$$

The orthogonal projection $P = P_E$ onto E is defined as

$$Px = e.$$

Generally, an orthog. proj is an operator $P \in \mathcal{L}(H,H)$ with $H = \operatorname{Im} P \oplus \operatorname{Ker} P$ being an orthog. decomposition.

Properties:

1) $1 - P$ is the orthogonal proj onto F.

2) $\operatorname{Im} P = E$, $\operatorname{Ker} P = F$.

3) $P^2 = P$. $\quad \big(P(\underbrace{Px}_{\in E}) = Px \big)$

4) $P^* = P$ (selfadjoint)

(indeed, $\langle Px, y \rangle = \langle Px, (1-P)y + Py \rangle$
$= \langle Px, Py \rangle$ because $\underset{\in E}{Px} \perp \underset{\in F}{(1-P)y}$

Similarly $= \langle x, Py \rangle$.)

Exercise Let $P \in \mathcal{L}(H,H)$ be a selfadj. op. s.t. $P^2 = P$. Then P is an orthoprojection

~~5) $0 \le P \le I$ (positive); hence $\|P\| = 1$.~~

5) $\|P\| = 1$

(indeed, $\|P\| = \|P^2\| \le \|P\|^2 \Rightarrow \|P\| \ge 1$.
On the other hand, $x = Px + (1-P)x$ is an orthog. decomposition $\Rightarrow$
$\|x\|^2 = \|Px\|^2 + \|(1-P)x\|^2 \Rightarrow \|Px\|^2 \le \|x\|^2 \Rightarrow \|P\| \le 1$.)

6) $0 \leq P \leq I$.

$$\left(\begin{array}{l} \text{indeed: } \langle Px, x\rangle = \langle P^2 x, x\rangle = \langle Px, Px\rangle = \|Px\|^2 \geq 0 \qquad (*) \\ \text{But } \quad 0 \leq \|Px\|^2 \leq \|x\|^2 \quad \text{by Property(5)} \\ \qquad\qquad\qquad\quad = \langle x, x\rangle . \end{array} \right)$$

QED.

Prop (majorization, range inclusion, factorization)

Let P_1, P_2 be orthogonal projections in H. TFAE:

(i) $P_1 \leq P_2$

(ii) $\operatorname{Im} P_1 \subseteq \operatorname{Im} P_2$

(iii) $P_1 P_2 = P_1$

Proof

(i) ⇒ (ii). Since $\langle P_1 x, x\rangle = \|P_1 x\|^2$ and $\langle P_2 x, x\rangle = \|P_2 x\|^2$ by $(*)$,

(i) states that $\|P_1 x\| \leq \|P_2 x\|$ for all $x \in H$.

This clearly implies $\operatorname{Ker} P_1 \supseteq \operatorname{Ker} P_2$. ~~Since $\operatorname{Im} P_i = (\operatorname{Ker} P_i)^\perp \Rightarrow$~~

Passing to the orthogonal complements, we conclude that $\operatorname{Im} P_1 \subseteq \operatorname{Im} P_2$.

(ii) ⇒ (iii) The identity $P_1(1 - P_2) = 0$ follows from the inclusion

$\operatorname{Im}(1 - P_2) = (\operatorname{Im} P_2)^\perp \subseteq (\operatorname{Im} P_1)^\perp = \operatorname{Ker} P_1$. ~~QED~~

(iii) ⇒ (ii) It suffices to check that $P_2 - P_1$ is an orthogonal projection. ~~since as we know~~

$$(P_2 - P_1)^2 = (P_2 - P_1)(P_2 - P_1) = \underbrace{P_2^2}_{= P_2} - \underbrace{P_1 P_2}_{= P_1} - \underbrace{P_2 P_1}_{= P_1^* = P_1} + \underbrace{P_1^2}_{= P_1} = P_2 - P_1.$$

QED.

A version of Proposition P.152 holds for general ~~bounded~~ linear operators on H.

This is known as R. Douglas' lemma (see below).

We will need one part of it:

$T_1T_2 = T_2T_1$

Prop Assume that $0 \le T_1 \le T_2$ for some (self adjoint) commuting T_1, T_2 on H.

Then $\|T_1 x\| \le \|T_2 x\|$ for all $x \in H$.

~~In particular~~.

Consequently, $\mathrm{Ker}\, T_1 \supseteq \mathrm{Ker}\, T_2$

~~Remark. The conclusion $\Leftrightarrow (\mathrm{Im}\,T_1)^\perp \supseteq (\mathrm{Im}\,T_2)^\perp \Rightarrow \mathrm{Im}\,T_1 \subseteq \mathrm{Im}\,T_2$. For projections whose images are closed [illegible] gives (ii) in Prop 152~~

For the proof we need a (useful) lemma

Remark: Conclusion $\Leftrightarrow (\mathrm{Im}\,T_1)^\perp \supseteq (\mathrm{Im}\,T_2)^\perp$
$\Leftrightarrow \overline{\mathrm{Im}\,T_1} \le \overline{\mathrm{Im}\,T_2}$
(Compare to Prop. 152)

Lemma ~~Let S, T be positive self adj. operators on H which commute~~

Let S, T be commuting self-adj operators on H ~~(i.e. $ST = TS$)~~

If $S, T \ge 0$ then $ST \ge 0$

Move this lemma to

Proof

$$\langle STx, x\rangle = \langle \sqrt{S}\sqrt{S}\sqrt{T}\sqrt{T}\,x, x\rangle$$
$$= \langle \sqrt{T}\sqrt{S}\sqrt{S}\sqrt{T}\,x, x\rangle$$

(f(t) $\sqrt{S}$ and f(t) $\sqrt{T}$ commute because S, T commute)

$$= \langle \sqrt{S}\sqrt{T}x, \sqrt{S}\sqrt{T}x\rangle \ge 0.$$

QED

Proof of Prop. $\|T_1 x\|^2 = \langle T_1 x, T_1 x\rangle = \langle T_1^2 x, x\rangle$, and similarly

$\|T_2 x\|^2 = \langle T_2^2 x, x\rangle$.

To complete the proof, it suffices to show that $T_1^2 \le T_2^2$

$T_2^2 - T_1^2 = (T_2 - T_1)(T_2 + T_1)$ because T_1, T_2 commute.

Both terms are positive by assumption.

~~$T_2 - T_1 \ge 0$ by assumption~~. Their product is then positive by lemma. QED

A more general result is Douglas' lemma ($T_1T_1^* \le T_2T_2^* \Leftrightarrow \mathrm{Im}\,T_1 \le \mathrm{Im}\,T_2 \Leftrightarrow T_1 = ST_2$)

Resolutions of identity. (A continuous version of orthogonal decompositions)

Def A family of projections (P_λ) indexed by $\lambda \in \mathbb{R}$ is called a resolution of identity if ~~there exists an interval $[m, M]$ s.t~~

(i) P_λ is increasing in λ, i.e. $P_\lambda \le P_\mu$ for $\lambda \le \mu$;

(ii) There exists some interval $[m, M]$ such that

$$P_\lambda = 0 \text{ for } \lambda < m \quad \text{and} \quad P_\lambda = I \text{ for } \lambda \ge M$$

(iii) P_λ is right-continuous, i.e.

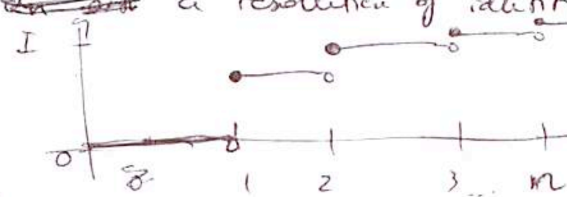
$$\lim_{\lambda \to \lambda_0^+} P_\lambda x = P_{\lambda_0} x \quad \text{for all } \lambda_0 \in \mathbb{R},\ x \in H.$$

~~This is~~ We will call this a resolution of identity on $[m, M]$.

Examples.

1) Suppose $H_1 \oplus H_2 \cdots \oplus H_n = H$ be an orthogonal decomposition & let Q_i be the projections onto H_i.

Then $P_\lambda = \sum_{i \le \lambda} Q_i$ is a resolution of identity



Discontinuities at 1, 2, 3 ..

2) let $T = \sum_{i=1}^{\infty} \lambda_i Q_i$ be the spectral representation of a compact self-adjoint op. T.

Here λ_i are eigenvectors and Q_i are orthogonal proj. onto the eigenspace (of eigenvalues) corr. to λ_i

$P_\lambda = \sum_{\lambda_i \le \lambda} Q_i$ is a resolution of identity. P_λ are called the "spectral proj's"

Discontinuities at λ_i's.

3) In $L_2[0,1]$, $P_\lambda(f) := f \cdot \mathbb{1}_{[0,\lambda]}$ defines ~~a spectral~~ a resolution of identity. No discontinuities.