LECTURE 37

Spectral projections — following [C. W. Groetsch]

- ~~...~~ We begin to develop a spectral theory of ~~...~~ general self-adjoint operators T on $\mathcal{H}$ (not necessarily compact!)

- ~~There may be a continuum of spectral points. The sum~~

- Recall the spectral thm for self-adj. compact:

$$T = \sum_{i=1}^{\infty} \lambda_i Q_i$$

where λ_i are eigenvalues of T, Q_i are the orthog. proj's onto eigenspaces

~~...~~

- For non-compact T, there may be a continuum of spectrum points. The sum $\longrightarrow$ integral. $\boxed{T = \int \lambda \, dP_\lambda}$ where ~~P~~ dP_λ is the orthog. projection onto ~~an infinitesimal~~ the "eigenspace" corresp. to λ, even though there may be no eigenvectors!

~~...~~ • Recall that $P_\lambda = \sum_{\lambda_i \le \lambda} Q_i$ are called the spectral proj's.

The family (P_λ) is a resolution of identity.

~~• A contin~~

- We are interested in a general (~~continuous~~ non-compact) ~~version of this~~ for general (noncompact) T.

i.e., we want to define ~~...~~ P_λ an orthog. ~~projection~~ corresponding ~~to ... eigenvalue~~ to the spectrum pts $\le \lambda$.

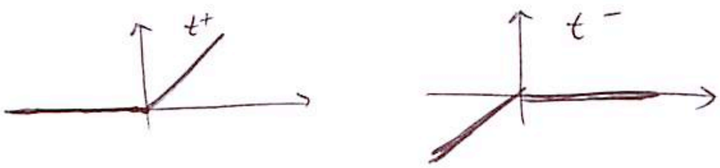
• Instea

Partial sums $\sum_{\lambda_i \le \lambda} \lambda_i Q_i \Rightarrow$?

- How to extract from T the part corresponding to eigenvalues $\le \lambda$?

• We zero-out the eigenvalues $> \lambda$ by applying an appropriate function to T.

• Define $f(t) = t^+$ and $g(t) = t^-$:



$$t^+ = \frac{t + |t|}{2} \geq 0; \quad t^- = \frac{t - |t|}{2} \leq 0$$

$$t^+ + t^- = t, \quad t^+ t^- = 0.$$

• Functional calculus:

$$f(T) = T^+ ; \quad g(T) = T^-.$$

Same properties for T^+, T^-.

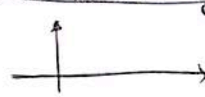
• Examples: 1) ~~Diag. operator on~~ Diag. operator on ℓ_2: $T = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix}$; $T^- = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & 0 & \\ 0 & & & 0 \ddots \end{pmatrix} \leq 0$, $T^+ = \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4 \ddots \end{pmatrix} \geq 0$

2) More generally, Spectral repres. $T = \sum \lambda_k Q_k$

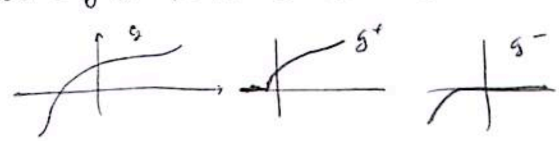
$$T^+ = \sum_{\lambda_k \geq 0} \lambda_k Q_k ; \quad T^- = \sum_{\lambda_k \leq 0} \lambda_k Q_k$$

3) Multiplication operator in $L_2[0,1]$. $(Tf)(t) = g(t) f(t)$.

~~T^+ is a mult. by g^+; T^- is a mult. by g^-~~.



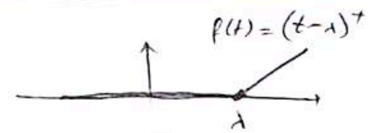
$(T^+ f)(t) = g^+(t) f(t)$; $(T^- f)(t) = g^-(t) f(t)$.



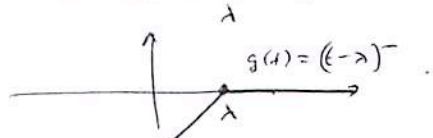
~~Shifting $T_\lambda = (T - \lambda I$; $T_\lambda^+ =$~~

• Shifting Def: $\boxed{T_\lambda = T - \lambda I ; \quad T_\lambda^+ = (T - \lambda I)^+ ; \quad T_\lambda^- = (T - \lambda I)^-}$

Equivalently, $T_\lambda^+ = f(T)$ for



$T_\lambda^- = g(T)$ for



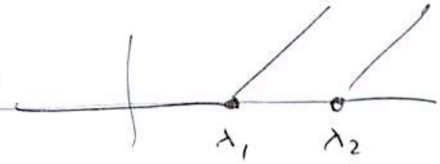
Examples: 2) $T = \sum \lambda_i Q_i$

$$T_\lambda^+ = \sum_{\lambda_i > \lambda} (\lambda_i - \lambda) Q_i$$

1) $T = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix}$ 2) $T^+ = \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ & & \lambda_3 - \lambda & \\ 0 & & & \lambda_4 - \lambda \end{pmatrix} \geq 0$

$T^- = \begin{pmatrix} \lambda_1 - \lambda & & & \\ & \lambda_2 - \lambda & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \leq 0$.

~~Proof~~ Clearly, $(t-\lambda_1)^+ \geq (t-\lambda_2)^+$ if $\lambda_1 \leq \lambda_2$



~~Hence T~~ By Cor. p.146,

$$(T-\lambda_1 1)^+ \geq (T-\lambda_2 1)^+ .$$

Similarly for "−".

We have proved:

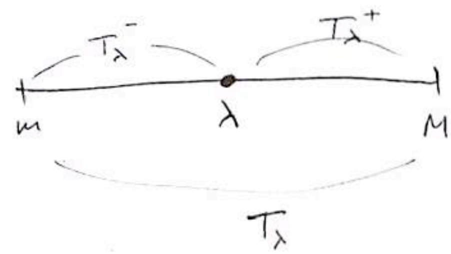
Prop $\boxed{\{T_\lambda^+\} \text{ is decreasing in } \lambda; \quad T_\lambda^- \text{ is increasing in } \lambda.}$ ∎

Let $[m, M]$ be the tightest interval containing $\sigma(T)$, i.e.

$$m = \inf_{x\in S_H} \langle Tx, x\rangle, \quad M = \sup_{x \in S_H} \langle Tx, x\rangle$$

(see Thm p.132)

Prop If $\lambda \leq m$ then $T_\lambda^- = 0$, $T_\lambda^+ = T_\lambda$.
If $\lambda \geq M$ then $T_\lambda^- = T_\lambda$, $T_\lambda^+ = 0$.



Proof By def's of m, M,

$$mI \leq T \leq MI$$

Hence for $\lambda \leq m$, ~~$T-\lambda 1 \geq 0$ and thus $T_\lambda^- = (T-\lambda 1)^- = 0$,~~

$T_\lambda^+ = (T-\lambda 1)^+ = T$.

$T - \lambda I \geq 0$.

Since the function $(t-\lambda)^- = 0$ for $t - \lambda \geq 0$,

the operator $T_\lambda^- = (T-\lambda I)^- = 0$. (see Cor. p.146).

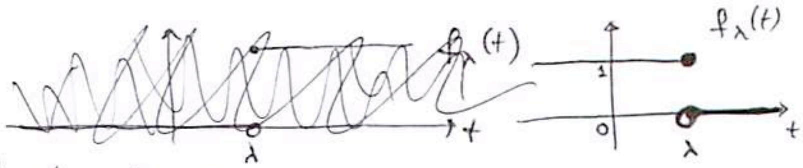
~~Similarly~~ One can prove the other assertions similarly. QED

(Ex: do this!)

Def (Spectral projections)

The orthogonal projections in H onto $\mathrm{Ker}(T_\lambda^+)$ are called the spectral projections of T, and denoted P_λ.

• Alternatively, $P_\lambda = f_\lambda(T)$ for



However, we have not defined functional calculus for discontinuous functions. (This is possible, too!)

~~The spectral projection P_λ~~

Examples i) $T = \sum \lambda_i Q_i$

Recall $T_\lambda^+ = \sum_{\lambda_i > \lambda} (\lambda_i - \lambda) Q_i \Rightarrow \ker(T_\lambda^+) = \{$span of the eigenvectors corresp. to $\lambda_i \le \lambda\}$

$$\Rightarrow \boxed{P_\lambda = \sum_{\lambda_i \le \lambda} Q_i}$$

Spectral projections form a resolution of identity

~~Property 1~~ The first two defining properties of the res of id (p. 154) are simple:

~~Lemma (first two proper~~ Let (m, M) be the spectral interval as in p. 157.

Prop (i) P_λ is increasing in λ

(ii) $P_\lambda = 0$ for $\lambda \le m$ and $P_\lambda = I$ for $\lambda \ge M$

Proof (i) T_λ^+ is decreasing in λ (Prop p.157)

Hence $\ker T_\lambda^+$ is increasing in λ (by Prop. p. 153)

Hence P_λ (the proj. onto $\ker T_\lambda^+$) is increasing in λ (by prop. p. 152)

(ii) For $\lambda \le m$, ~~$T_\lambda^+ = T_\lambda$ (by prop. p. 157)~~ λ is a regular point (recall Thm p. 132)

Hence $0 = \ker T_\lambda = \ker T_\lambda^+$ (by Prop p. 157) $\Rightarrow P_\lambda = 0$.

Similarly for $\lambda \ge M$

QED