Prop $\boxed{\text{The spectral projections } P_\lambda \text{ commute with } T.}$

Proof follows from

Exercise Let $T \in L(H,H)$ and let P be an orthogonal projection in H.
Then T and P commute iff $\ker P$, $\operatorname{Im} P$ are invariant subspaces of T.
In particular, if T is selfadjoint then T, P commute iff $\operatorname{Im} P$ is invariant. (the latter follows from Prop p.135)

Claim $\operatorname{Im} P_\lambda = \ker(T_\lambda^+)$ is an invariant subspace for both T_λ^+, T_λ^-

Indeed, let $x \in \ker T_\lambda^+$. Then $T_\lambda^+ x = 0 \in \ker(T_\lambda^+)$.

$$T_\lambda^+(T_\lambda^- x) = T_\lambda^-\underbrace{(T_\lambda^+ x)}_{=0} = 0$$

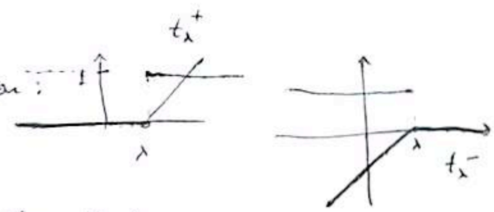
(because T_λ^+, T_λ^- commute as functions of T – in fact, $T_\lambda^+ T_\lambda^- = T_\lambda^- T_\lambda^+ = 0$ as $t_\lambda^+ t_\lambda^- = t_\lambda^- t_\lambda^+ = 0$)

Hence $T_\lambda^- x \in \ker T_\lambda^+$.

• It follows that T_λ^+, T_λ^- commute with P; so does $T_\lambda = \frac{T_\lambda^+ + T_\lambda^-}{2}$. QED.

Cor $\boxed{T_\lambda^+ = T_\lambda(1-P_\lambda) \; ; \quad T_\lambda^- = T_\lambda P_\lambda}$

Illustration:



Proof ~~First note that~~ By Prop. above, it suffices to show that

$$T_\lambda^+ = (1-P_\lambda)T_\lambda \; ; \quad T_\lambda^- = P_\lambda T_\lambda$$

First note that
$$P_\lambda T_\lambda^+ = 0 \quad \text{as} \quad \operatorname{Im}(T_\lambda^+) \subseteq \ker(T_\lambda^+)^\perp \quad (\text{as selfadj.})$$
$$= \ker P_\lambda$$

Also,
$$P_\lambda T_\lambda^- = T_\lambda^- \quad \text{as} \quad T_\lambda^+ T_\lambda^- = 0, \text{ thus } \operatorname{Im}(T_\lambda^-) \subseteq \ker(T_\lambda^+) = \operatorname{Im} P_\lambda$$

Therefore,
$$(1-P_\lambda)T_\lambda = \underbrace{(1-P_\lambda)T_\lambda^+} + \underbrace{(1-P_\lambda)T_\lambda^-} = T_\lambda^+$$

and
$$P_\lambda T_\lambda = P_\lambda T_\lambda^+ + P_\lambda T_\lambda^- = T_\lambda^-.$$

QED

Lemma (localization) For $\lambda_1 \le \lambda_2$,

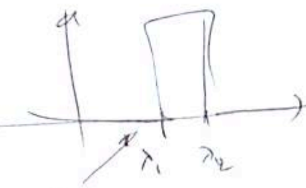
$$\boxed{\lambda_1 (P_{\lambda_2} - P_{\lambda_1}) \le T(P_{\lambda_2} - P_{\lambda_1}) \le \lambda_2 (P_{\lambda_2} - P_{\lambda_1})}$$

Proof LHS $\Leftrightarrow$ $T_{\lambda_1}(P_{\lambda_2} - P_{\lambda_1}) \geq 0$?

$$\begin{aligned} &T_{\lambda_1}(P_{\lambda_2} - P_{\lambda_1}) \\ &= T_{\lambda_1}(1 - P_{\lambda_1})(P_{\lambda_2} - P_{\lambda_1}) \quad \text{because } P_{\lambda_1}(P_{\lambda_2} - P_{\lambda_1}) = P_{\lambda_1} - P_{\lambda_1} = 0 \\ &= T_{\lambda_1}^{+}(P_{\lambda_2} - P_{\lambda_1}) \quad \cancel{\text{because } T_{\lambda_1}} \text{ by Cor p.159} \\ &\geq 0. \end{aligned}$$

QED

Similarly, RHS.



- Interpretation. $P_{\lambda_2} - P_{\lambda_1}$ is an orthoprojection; ~~which~~ it commutes with T. Intuitively, onto the eigenvectors corr. to $(\lambda_1, \lambda_2]$.

Hence $H_{\lambda_1 \lambda_2} := \operatorname{Im}(P_{\lambda_2} - P_{\lambda_1})$ is an invariant subspace of T.

~~It~~ Localization lemma yields: for $x \in H_{\lambda_1 \lambda_2}$,

$$\lambda_1 \langle x, x \rangle \le \langle Tx, x \rangle \le \lambda_2 \langle x, x \rangle$$

$$\Rightarrow \quad \boxed{\left\| (T - \lambda 1)\big|_{H_{\lambda_1 \lambda_2}} \right\| \le |\lambda_2 - \lambda_1|} \leftarrow \text{small}$$

"approx. "eigenspace", but may be empty

~~Lemma (P_λ) form a resolution of~~

Lemma $\boxed{P_\lambda \text{ is right-continuous}}$

Proof We have to show ~~that~~

$$\lim_{\lambda \to \lambda_0^+} P_\lambda x = P_{\lambda_0} x \qquad \forall x$$

By Monotone convergence Thm. (HW), a limit exists: $\exists\ A \in \mathcal{L}(H, H)$:

$$\lim_{\lambda \to \lambda_0^+} P_\lambda x = Ax.$$

We have to show that $P_{\lambda_0} = A$.

By Localization Lemma,

$$\lambda_0 (P_\lambda - P_{\lambda_0}) \le T(P_\lambda - P_{\lambda_0}) \le \lambda (E_\lambda - P_{\lambda_0})$$

$$\Big\downarrow \lambda \to \lambda_0$$

$$\lambda_0 (A - P_{\lambda_0}) \le T(A - P_{\lambda_0}) \le \lambda_0 (A - P_{\lambda_0})$$

$$\Downarrow$$

$$T_{\lambda_0}(A - P_{\lambda_0}) = 0$$

~~$\Rightarrow \operatorname{Im}(A - E_{\lambda_0}) = \ker T_\lambda$~~

Since $T_{\lambda_0}^+ = (1 - P_{\lambda_0}) T_{\lambda_0} \quad \Rightarrow$

$$T_{\lambda_0}^+ (A - P_{\lambda_0}) = 0.$$

$$\Rightarrow \operatorname{Im}(A - P_{\lambda_0}) \subseteq \ker T_{\lambda_0}^+$$

P_{λ_0} projects onto here

~~$\Rightarrow P_{\lambda_0}(A - E_{\lambda_0}) = A - E_{\lambda_0}$~~ $\Rightarrow A - P_{\lambda_0} = P_{\lambda_0}(A - P_{\lambda_0})$

$$= \lim_{\lambda \to \lambda_0^+} P_{\lambda_0}(P_\lambda - P_{\lambda_0}) = 0 \qquad (\text{as } P_{\lambda_0} P_\lambda = P_{\lambda_0}).$$

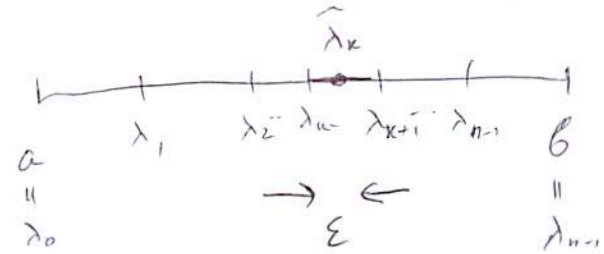
QED

We have proved:

$\boxed{\text{THM The spectral proj's } (P_\lambda) \text{ form a resolution of identity.}}$

Spectral integral

(Riemann-Stiltjes)



$$I = \sum_{k=0}^{n-1} \left(P_{\lambda_{k+1}} - P_{\lambda_k} \right)$$

$$\Rightarrow T = \sum_{k=0}^{n-1} T \left(P_{\lambda_{k+1}} - P_{\lambda_k} \right)$$

By the localization lemma,

$$\underbrace{\sum_{k=0}^{n-1} \lambda_k \left(P_{\lambda_{k+1}} - P_{\lambda_k} \right)}_{= LHS} \leq T \leq \underbrace{\sum_{k=0}^{n-1} \lambda_{k+1} \left(P_{\lambda_{k+1}} - P_{\lambda_k} \right)}_{= RHS}$$

This is a good approximation because

$$RHS - LHS = \sum_{k=0}^{n-1} \underbrace{(\lambda_{k+1} - \lambda_k)}_{\in [0, \varepsilon]} \left(P_{\lambda_{k+1}} - P_{\lambda_{k-1}} \right)$$

$$\Rightarrow \quad 0 \leq RHS - LHS \leq \varepsilon I$$

$$\Rightarrow \quad \| RHS - LHS \| \leq \varepsilon.$$

We have shown that $\forall \tilde{\lambda}_k \in (\lambda_k, \lambda_{k+1})$,

$$\left\| T - \sum_{k=0}^{n-1} \tilde{\lambda}_k \left(P_{\lambda_{k+1}} - P_{\lambda_k} \right) \right\| \leq \varepsilon.$$

$\Rightarrow$ Riemann-Stiltjes integral

$$\boxed{T = \int_{-\infty}^{\infty} \lambda \, dP_\lambda}$$

__Example__ : discrete case $\quad T = \sum_i \lambda_i Q_i$

Summarizing, we have shown ;

Spectral Thm (Hilbert) ~~Let $T \in L(H,H)$~~

Let T be a bounded self-adjoint op. on a Hilbert space H,

$$m = \inf_{x \in S_H} \langle Tx, x\rangle \;; \quad M = \sup_{x \in S_H} \langle Tx, x\rangle .$$

~~There is a spectral~~ ~~representation~~

There exists a resolution of identity (P_λ) on $[m, M]$ such that i) all P_λ commute with $\forall$ op that commutes with T,

$$ii) \quad T = \int_m^M \lambda \, dP_\lambda .$$

Cor ~~$\|T\| = \int_m^M \|$~~ $\|Tx\|^2 = \int_m^M \lambda^2 \, d\langle P_\lambda x, x\rangle$ (Ex: prove, using the orthogonality of dP_λ for different λ)

Additional properties ; (see [EMT]) ;

~~1. λ is in the point spectrum~~

THM (Spectrum).

1. λ_0 is a regular point iff (P_λ) is constant in some neighborhood of λ_0
2. λ_0 is in the point spectrum iff λ_0 is a point of discontinuity of (P_λ).
3. Otherwise λ_0 is in the continuous spectrum, i.e. (P_λ) is not const but continuous at λ_0.

THM (Functional calculus) For a continuous function f,

$$f(T) = \int_m^M f(\lambda) \, dP_\lambda .$$

[Groetsch]