LECTURE 4

Example: l_p = space of all p-summable sequences.

let $1 \le p < \infty$.

l_p consists of all sequences $x = (x_i)_1^\infty$ satisfying

$$\|x\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{1/p} < \infty$$

Note l_p is a partial case of $L_p(\Omega, \Sigma, \mu)$ where $\Omega = \mathbb{N}$, μ = counting measure.

Corollary (Minkowski inequality for seq). $\forall\, 1 \le p < \infty$, $\forall$ two sequences (a_i) and (b_i) of numbers (finite or infinite)

$$\left(\sum_i |a_i + b_i|^p \right)^{1/p} \le \left(\sum_i |a_i|^p \right)^{1/p} + \left(\sum_i |b_i|^p \right)^{1/p}$$

i.e. $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

~~Minkowski functional~~

~~How to construct convex functions (and therefore norms) from convex sets.~~

~~Let $K \subset E$ linear space. be a convex set in $\mathbb{R}^n$ (for simplicity)~~

~~Proposition Let K be a closed~~

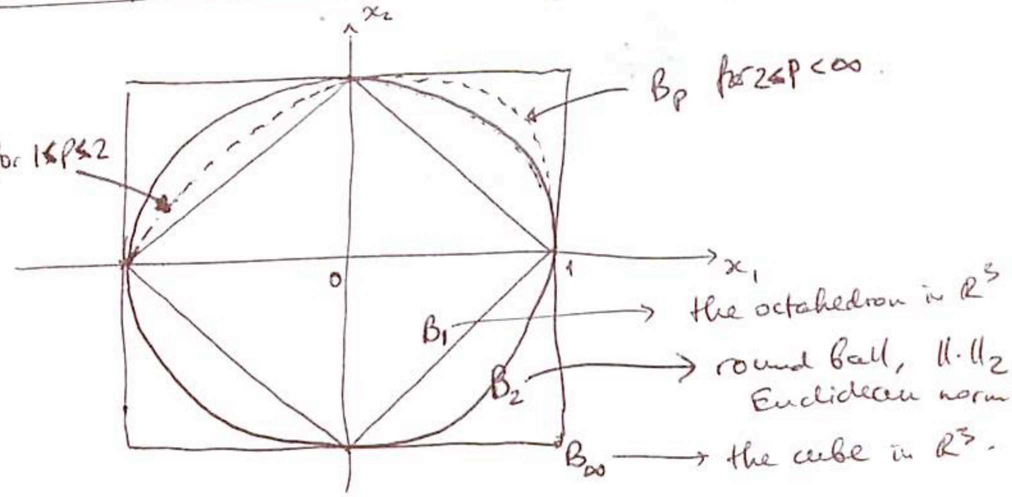
Example ℓ_p^n = ~~space of all p-summable~~ n-dimensional version of ℓ_p.

ℓ_p^n consists of vectors $x = (x_i)_1^n$ with the norm

$$\|x\|_p := \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} \text{ for } 1 \le p < \infty, \qquad \|x\|_\infty := \max_{1\le i\le n} |x_i| .$$

Therefore, $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$.

Illustration of the unit balls B_p of ℓ_p^n for $n=2$.



Exercise (HW): For every $x \in \mathbb{R}^n$,

$$\|x\|_p \to \|x\|_\infty \quad \text{as } p \to \infty .$$

This explains the "∞" index in $\|\cdot\|_\infty$ norm.

~~Exercise (Minkowski functional~~

Exercise (Minkowski functional) – see HW

Subspaces and quotient spaces of normed spaces

As ~~comm~~ is common in topology:

Def A linear subspace Y of X with the norm induced by X is called a subspace of X.

0) The space of all polynomials $\mathcal{P}$ is a dense subspace of $C[0,1]$ (Weierstrass)

Examples 1) $C[0,1]$ is a dense subspace of $L_1[0,1]$

[every continuous function on $[0,1]$ is integrable;
$\forall \varepsilon$, $\forall$ every integrable function f $\exists$
$\exists$ a continuous function g s.t. $\int_0^1 |f-g|\, d\mu < \varepsilon$]
— see Wikipedia on L_p spaces

2) c_0 is a closed subspace of ℓ_∞ (HW) (~~Exercise moved to HW~~)

3) Similarly, ℓ_p is a closed subspace of ℓ_∞ ~~($p<\infty$)~~ ($\forall p$.)
but ℓ_p is a dense subspace of c_0. (Exercise)

Remark Those two types of subspaces, dense and closed, are most frequently encountered.

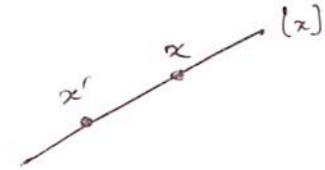
Def (Quotient spaces) Let Y be a closed subspace of a normed space X. For every coset $[x] = x + Y$, we define the norm on X/Y by

$$\|[x]\| := \inf_{y \in Y} \|x+y\|.$$

In other words, $\|[x]\| = \text{dist}(0, [x])$.

Proposition Def. above indeeds defines a norm on X/Y. ~~The norm on X/Y is well defined~~

1. Well defined, i.e. independent of a choice of x in the coset $[x]$: if ~~$x_1, x_2 \in$~~ $x' \in [x]$



~~then~~ ~~inf~~ then $x - x' \in Y \Rightarrow$

$$\inf_{y \in Y} \|x + y\| = \inf_{y \in Y} \|x' + \underbrace{(x - x') + y}_{y' \in Y}\| = \inf_{y' \in Y} \|x' + y'\| .$$

2. Norm axioms:

(i) Let $\|[x]\| = 0$. Then $\inf_{y \in Y} \|x - y\| = 0$.

Hence x is a limit point of Y.

But Y is closed $\Rightarrow x \in Y \Rightarrow [x] = 0$.

(ii) $\|[\lambda x]\| = \inf_{y \in Y} \|\lambda x + y\| = \inf_{y \in Y} \|\lambda x + \lambda y\| = \lambda \inf_{y \in Y} \|x + y\| = \lambda \|[x]\|$.

(iii) $\|[x_1 + x_2]\| \overset{?}{\leq} \|[x_1]\| + \|[x_2]\|$

Let $\varepsilon > 0$. $\exists\, y_1, y_2 \in Y$: $\|[x_1]\| \geq \|x_1 + y_1\| - \varepsilon$,
$\|[x_2]\| \geq \|x_2 + y_2\| - \varepsilon$.

~~$\Rightarrow \|[x_1 + x_2]\| =$~~ $\Rightarrow \|x_1 + x_2 + y_1 + y_2\| \leq \|x_1 + y_1\| + \|x_2 + y_2\|$ by Δ ineq. in X

$\leq \|[x_1]\| + \|[x_2]\| + 2\varepsilon$.

~~Taking inf~~ ~~Since $y_1 + y_2 \in Y \Rightarrow$~~

~~$\|[x_1 + x_2]\|$~~ $= \inf$

$\Rightarrow \|[x_1 + x_2]\| = \inf_{y \in Y} \|x_1 + x_2 + y\| \leq \|x_1 + x_2 + y_1 + y_2\| \leq \|[x_1]\| + \|[x_2]\| + 2\varepsilon$.

$\varepsilon \to 0 \Rightarrow$ QED.

Example: L_∞. As usual, (Ω, Σ, μ) denotes a measure space.

$\mathcal{L}_\infty :=$ the space of all ~~bounded~~ measurable functions $f: \Omega \to \mathbb{R}$,

with the norm

$$\|f\|_\infty := \sup_{t \in \Omega} |f(t)| .$$

Exercise: check the norm axioms.

Shortcoming of $\mathcal{L}_\infty$: functions that are $= 0$ a.e. may have norms $\neq 0$,

e.g. $f =$ .

Solution: consider a subspace

$Y := \{$all functions in $\mathcal{L}_\infty$ that are $= 0$ a.e.$\}$

and define

$$L_\infty := \mathcal{L}_\infty / Y.$$

By Prop. on 17, this is a normed space. In fact, we needed:

~~But~~

Proposition: Y is a closed subspace of $\mathcal{L}_\infty$.

~~$f_n \to f$ ... Let $f_n \in Y$, $f_n \to f$ uniformly. Need to check~~

Need to check: if ($f_n = 0$ a.e., $f_n \to f$ uniformly) $\Rightarrow$ $f = 0$ a.e.

This can be shown as follows. Clearly, $|f_n| = 0$ a.e., $|f_n| \to |f|$ uniformly $\Rightarrow |f_n|$ is unif. bounded.

By the Dominated Convergence Thm,

$$\int |f| = \lim_n \int |f_n| = 0.$$

$\Rightarrow |f| = 0$ a.e. $\Rightarrow f = 0$ a.e.

Exercise: $C[0,1]$ is not dense in $L_\infty[0,1]$.

Example C^k, $1 \le k < \infty$

$C^k[a,b)$ consists of ~~all~~ the functions $f:(a,b) \to \mathbb{R}$ ~~st~~ which have derivatives up to the k'th order, with the norm

$$\cancel{\|f\| := \max_{t\in(a,b)} \max \left(|f'(x) \right.}$$

$$\|f\| = \max \left(\|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty, \dots, \|f^{(k)}\|_\infty \right)$$

Exercise : Check the norm axioms.

Example Sobolev space W^k_p, $1 \le k < \infty$, $1 \le p \le \infty$

$W^k[a,b)$ is the same linear space as $C^k(a,b)$ but with the different norm

$$\|f\| = \left(\sum_{j=1}^{k} \|f^{(j)}\|_p^p \right)^{1/p} = \left(\sum_{j=1}^{k} \int_a^b |f^{(j)}|^p \, d\mu \right)^{1/p}$$

Generally, for a domain $D \subset \mathbb{R}$ or $\mathbb{R}^n$, $W^k(a,b) \neq C^k(a,b)$ as a linear space (cont's functions may be non-integrable)