Banach spaces.

Def: A complete normed space is called a Banach space.

Recall the notion of completeness (studied before in metric spaces).

- A seq. \((x_n)\) in a normed space is called Cauchy if
  \[ \|x_n - x_m\| \to 0 \text{ as } n, m \to \infty \]
  i.e. \(\exists \varepsilon > 0 \in \mathbb{R} \) s.t.
  \[ \|x_n - x_m\| < \varepsilon \quad \forall n, m \in \mathbb{N} \]
- A normed space \(X\) is complete if a Cauchy sequence converges to a vector in \(X\).

Example:
1. \(\mathbb{R}\) is a complete space (analysis course).
2. \(\mathbb{Q}\) is not.

THM: \(C(K)\) is a Banach space, where as usual \(K\) is a compact topological space.

Consider a Cauchy sequence \((f_n)\) in \(C(K)\):
\[ \|f_n - f_m\| \to 0, \quad n, m \to \infty. \]

Hence for every \(t \in K\), \((f_n(t))\) is a Cauchy sequence in \(\mathbb{R}\).

By the completeness of \(\mathbb{R}\), \(f_n(t)\) converges to a number \(\tilde{f} \in \mathbb{R}\).

There exists function \(f(t)\) s.t.
\[ f_n(t) \to f(t) \quad \forall t. \]

We claim that \(f_n \to f\) uniformly, i.e.
\[ \|f_n - f\|_{\infty} \to 0. \]

If we prove this, this would complete the proof, because the limit of a uniformly convergent sequence of continuous functions is a continuous function (course in analysis).
Now we prove the claim. Indeed, (* p.20) means that \( \forall \epsilon > 0 \) s.t.
\[
|f_n(t) - f(t)| \leq \epsilon 
\]
for all \( n, m \geq N(\epsilon) \) and all \( t \in K \).

Sending \( m \to \infty \) we recover
\[
|f_n(t) - f(t)| \leq \epsilon 
\]
for all \( n \geq N(\epsilon) \) and all \( t \in K \).

This means \( \|f_n - f\|_\infty \to 0 \), \( \epsilon \)

\( \square \)

**Exercise**

\( \ell_\infty \) is a Banach space; \( L_\infty(K) \) is a Banach space.

(Along the same lines)

**Proposition**

A closed subspace of a Banach space is a Banach space.

Let \( Y \) be a closed subspace of \( X \).

Consider a Cauchy seq. \((y_n)\) in \( Y \).

Since \( Y \subseteq X \) and \( X \) is complete, \( y_n \to x \) for some \( x \in X \).

But \( Y \) is closed \( \implies x \in Y \).

**Corollary**

\( C_0 \) is a Banach space

\( C_0 \) is a closed subspace of \( \ell_\infty \) — see homework
Series in Banach spaces.

Let \((x_k)\) be a sequence in a normed space \(X\).

As in the analysis course, the partial sums are defined as

\[ S_n := \sum_{k=1}^{n} x_k \]

If the partial sums converge to a vector \(S_n \to x \in X\),

then we say that the series \(\sum_{k=1}^{\infty} x_k\) converges to \(x\).

\[ \sum_{k=1}^{\infty} x_k = x. \]

**Def.** A series \(\sum_{k=1}^{\infty} x_k\) is absolutely convergent if \(\sum_{k=1}^{\infty} \|x_k\| < \infty\).

**Thm.** (Completeness criterion)

A normed space \(X\) is a Banach space iff every absolutely convergent series in \(X\) converges in \(X\).

\((\Rightarrow)\) Let \(X\) be a Banach space, \(\sum_{k=1}^{\infty} \|x_k\| < \infty\). (\(*\))

WTS: \(\sum_{k=1}^{\infty} x_k\) converges.

By completeness, it suffices to show that the partial sums are Cauchy, i.e. \(\|S_n - S_m\| \to 0\) as \(n,m \to \infty\).

\[ \|S_n - S_m\| = \|\sum_{k=m+1}^{n} x_k\| \leq \sum_{k=m+1}^{n} \|x_k\| \to 0 \text{ by } (\ast) \]

QED.
Assume \( X \) is incomplete, we will construct a divergent series which is nevertheless absolutely convergent.

By incompleteness, \( \exists \) Cauchy sequence \((v_n)\) in \( X \) which diverges.

\[
\text{Then } \exists \text{ a subsequence } (w_n) \subseteq (v_n) \text{ which satisfies } \|w_2 - w_1\| = \frac{1}{2}, \|w_3 - w_2\| = \frac{1}{4}, \|w_4 - w_3\| = \frac{1}{8}, \ldots
\]

Thus the sequence \( x_1 = w_2 - w_1, x_2 = w_3 - w_2, \ldots \)
forms an absolutely convergent series:

\[
\sum_{n=1}^{\infty} \|x_n\| = \frac{1}{2} + \frac{1}{4} + \ldots \leq 1
\]

But the partial sums \( w_n \) of the series \( \sum x_n \) diverge.

**THM**

\( L^p \) is a Banach space, \( 1 \leq p < \infty \)

(\( L^\infty \) also, see p.21)

Let \((f_n)\) in \( L^p \) form an absolutely convergent series, i.e.

\[
\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty.
\]

By the previous theorem, it suffices to show that \( \sum_{n=1}^{\infty} f_n \) converges in \( L^p \).

1. Assume all \( f_n \geq 0 \) first.

Then we are going to use the Monotone Convergence Theorem.

By Minkowski inequality,

\[
\|\sum_{n=1}^{N} f_n\|_p \leq \sum_{n=1}^{N} \|f_n\|_p \leq M
\]

\[
\Rightarrow \int |\sum_{n=1}^{N} f_n|^p \, dy \to \int |\sum_{n=1}^{\infty} f_n|^p \, dy \quad \text{as } n \to \infty
\]

which exists a.e. (pointwise limit)
1) Assume first that all $f_n \geq 0$.

Then the partial sums $\sum_{i=1}^{n} f_i$ increase; their pointwise limit is $\sum f_k$ (may be $\infty$ somewhere).

But the partial sums are bounded by Minkowski inequality:

$$\left\| \sum_{i=1}^{n} f_i \right\|_p \leq \sum_{i=1}^{n} \left\| f_i \right\|_p \leq M,$$

i.e.

$$\int \left( \sum_{i=1}^{n} f_i \right)^p \,du \leq M^p.$$  

Increasing seq. of functions.

Hence, by Lebesgue Monotone Convergence Theorem,

$$\int \left( \sum_{i=1}^{n} f_i \right)^p \,du \to \int \left( \sum \limits_{i}^{\infty} f_i \right)^p \,du \quad \text{as \ } n \to \infty.$$  

Hence $\sum_{i=1}^{n} f_i \leq L^p$.

Finally, $\sum \limits_{i}^{\infty} f_i$ is not only pointwise limit convergent, but it converges in $L^p$:

$$\left\| \sum_{i=1}^{n} f_i \right\|_p \leq \sum_{i=1}^{n} \left\| f_i \right\|_p \to 0 \text{ as } n \to \infty \text{ by the assumption}.$$

2) General $f_n$ : modify the above argument by replacing $f_i$ by $|f_i|$ where appropriate (Ex).