Banach spaces.

Def A complete normed space is called a Banach space.

Recall the notion of completeness (studied before in metric spaces).

- A seq. $(x_i)_1^\infty$ in a normed space is called Cauchy if

$$\|x_n - x_m\| \to 0 \quad \text{as } n > m \to \infty$$

i.e. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$\|x_n - x_m\| < \varepsilon \qquad \forall\, n, m > N.$$

- A normed space X is complete if $\forall$ Cauchy sequence converges to a vector in X.

~~Remarks~~:

Examples: 1) $\mathbb{R}$ is a complete space (analysis course).
2) $\mathbb{Q}$ is not.

THM $C(K)$ is a Banach space, where as usual K is a compact Hausdorff topological space

Consider a Cauchy sequence (f_n) in $C(K)$:

$$\|f_n - f_m\|_\infty \to 0, \quad n, m \to \infty. \qquad (*)$$

Hence for every $t \in K$, $(f_n(t))$ is a Cauchy sequence in $\mathbb{R}$.

By the completeness of $\mathbb{R}$, $f_n(t)$ converges to a number $\in \mathbb{R}$.

$\Rightarrow \exists$ function $f(t)$ s.t. $f_n \to f$ pointwise, i.e.

$$f_n(t) \to f(t) \qquad \forall t.$$

We claim that $f_n \to f$ uniformly, i.e. $\|f_n - f\|_\infty \to 0$.

~~If we prove this~~ This would complete the proof, because the limit of a uniformly convergent sequence of continuous functions is a continuous function (course in analysis)

Now we prove the claim.
~~Uniformly~~ (* p.20) means that $\forall \varepsilon \; \exists N(\varepsilon)$ s.t

$$|f_n(t) - f_m(t)| \leq \varepsilon \quad \text{for all } n,m > N(\varepsilon) \text{ and all } t \in K.$$

Sending $m \to \infty$ we recover

$$|f_n(t) - f(t)| \leq \varepsilon \quad \text{for all } n > N(\varepsilon) \text{ and all } t \in K.$$

This means $\|f_n - f\|_\infty \to 0$,

QED

Exercise ℓ_∞ is a Banach space; L_∞ is a Banach space.

(along the same lines)

Proposition A closed subspace of a Banach space is a Banach space.

Let Y be a closed subspace of X.
Consider a Cauchy seq. (y_n) in Y.
Since $Y \subseteq X$ and X is complete, $y_n \to x$ for some $x \in X$.
But Y is closed $\Rightarrow x \in Y$.

Corollary c_0 is a Banach space

(c_0 is a closed subspace of ℓ_∞ – see homework)

Series in Banach spaces

Let (x_K) be a sequence in a normed space X

As in the analysis course, the partial sums of $\sum_1^\infty x_k$ are defined as

$$s_n := \sum_1^n x_k$$

If the partial sums ~~converge to a vector~~ $S_n \to x \in X$ then we say that the series $\sum_1^\infty x_k$ converges to x:

$$\sum_1^\infty x_k = x.$$

Def A series $\sum_1^\infty x_k$ is absolutely convergent if $\sum_1^\infty \|x_k\| < \infty$.

THM (Completeness criterion)

A normed space X is a Banach space iff every absolutely convergent series in X converges in X.

($\Rightarrow$). Let X be a Banach space, $\sum_1^\infty \|x_k\| < \infty$. $(*)$

WTS: $\sum_1^\infty x_k$ converges.

~~Cons~~ By completeness, it suffices to show that the partial sums are Cauchy, i.e. $\|S_n - S_m\| \to 0$ as $n, m \to \infty$

$$\|S_n - S_m\| = \Big\| \sum_{n<k\le m} x_k \Big\| \le \sum_{n<k\le m} \|x_k\| \to 0 \quad \text{by } (*)$$

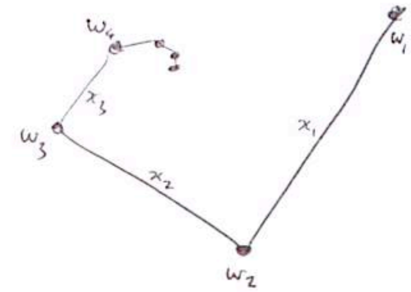
QED.

($\Leftarrow$) Assume X is incomplete; we will construct a divergent series which is nevertheless absolutely convergent.

By incompleteness, $\exists$ Cauchy sequence (v_n) in X which diverges.

Then $\exists$ a subsequence $(w_n) \subseteq (v_n)$ (still divergent), which satisfies

see HW $\left\{ \|w_2 - w_1\| \le \frac{1}{2}, \quad \|w_3 - w_2\| \le \frac{1}{4}, \quad \|w_4 - w_3\| \le \frac{1}{8}, \dots \right.$



Then the sequence $x_1 = w_2 - w_1$; $x_2 = w_3 - w_2, \dots$

forms an absolutely convergent series:

$$\sum_1^\infty \|x_k\| \le \frac{1}{2} + \frac{1}{4} + \dots \le 1.$$

But ~~its~~ the partial sums W_n of the series $\sum_1^\infty x_n$ diverge.

THM L_p is a Banach space, $1 \le p < \infty$ (L_∞ also, see p. 21)

Let (f_n) in L_p form an absolutely convergent series, i.e.

$$\sum_1^\infty \|f_k\|_p = M < \infty.$$

By the previous theorem, it suffices to show that $\sum_1^\infty f_k$ converges in L_p.

1) Assume all $f_k \ge 0$ first.

Then ~~we can use monotone~~ the partial sums of $\sum_1^\infty f_k$ increase,

(so we ~~can~~ are going to use the Monotone Convergence Theorem.)

By Minkowski inequality,

$$\left\| \sum_1^n f_k \right\|_p \le \sum_1^n \|f_k\|_p \le M$$

$$\overset{\text{M.C.T.}}{\Longrightarrow} \int \left(\sum_1^n f_k \right)^p d\mu \to \int \left(\sum_1^\infty f_k \right)^p d\mu \quad \text{as } n \to \infty$$

which exists a.e. (pointwise limit)

1) Assume first that all $f_k \geq 0$.

Then the partial sums $\sum_1^n f_k$ increase; their pointwise limit is $\sum_1^\infty f_k$ (may be ∞ somewhere).

But the partial sums are bounded by Minkowski inequality:

$$\left\| \sum_1^n f_k \right\|_p \leq \sum_1^n \|f_k\|_p \leq M ,$$

i.e.

$$\int \underbrace{\left(\sum_1^n f_k \right)^p}_{\uparrow \text{ increasing seq. of functions}} d\mu \leq M^p .$$

Hence, by Lebesgue Monotone Convergence Thm,

$$\underbrace{\int \left(\sum_1^n f_k \right)^p d\mu}_{\leq M^p} \longrightarrow \underbrace{\int \left(\sum_1^\infty f_k \right)^p d\mu}_{\leq M^p} .$$

$\Rightarrow$ hence in the limit,

Hence $\sum_1^\infty f_k \in L_p$.

Finally, $\sum_1^\infty f_k$ not only pointwise converges, but it converges in L_p:

$$\left\| \sum_n^\infty f_k \right\|_p \leq \sum_n^\infty \|f_k\|_p \to 0 \quad \text{as } n \to \infty \text{ by the assumption.}$$

2) General f_k: modify the above argument by replacing f_k by $|f_k|$ where appropriate (Ex.).