LECTURE 6

From the theory of metric spaces: (see (EMT) for normed spaces)

THM (Completion) Let X be a normed linear space.
There exists a normed space $\hat{X}$ (called completion of X) and a linear operator $T: X \to X'$ such that:

(i) $\|Tx\| = \|x\| \quad \forall x \in X$ (isometry into)

(ii) $\mathrm{Im}(T)$ is dense in $\hat{X}$.

The completion is unique up to isometry.

Example: $L_p(a,b)$ $(1 \le p < \infty)$ is a completion of $C(a,b)$ (~~also of $\mathcal{P}(x)$~~) in the L_p-norm.

Application of completeness

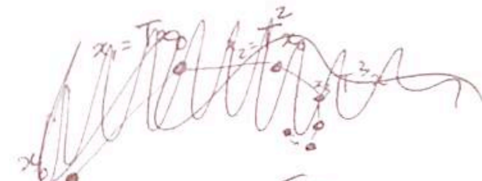
Thm (Fixed point of contractions) Let X be a Banach space.
Let $T: X \to X$ be a contraction, i.e. $\|Tx - Ty\| \le \alpha \|x-y\|$ for some $\alpha \in (0,1)$ and all $x,y \in X$. Then T has a unique fixed point $x \in X$, i.e. $T(x) = x$.

0) T is continuous (b/c contraction)

Let $x_0 \in X$ be arbitrary,

$x_n := Tx_{n-1} = T^n x, \quad n = 1, 2, \dots$

And: $\|x_n - x_{n+1}\| \le \alpha \|x_{n-1} - x_n\| \le \dots \le \alpha^n \|x_0 - x_1\|$



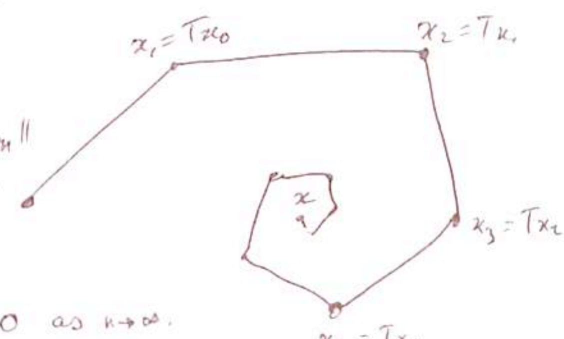
Redo using series (criterion of completeness)

1) (x_n) is Cauchy: if $m > n$,

$$\|x_n - x_m\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \dots + \|x_{m-1} - x_m\|$$

$$\le \left(\alpha^n + \alpha^{n+1} + \dots + \alpha^{m-1}\right)\|x_0 - x_1\|$$

$$= \frac{\alpha^n - \alpha^m}{1-\alpha}\|x_0 - x_1\| \le \frac{\alpha^n}{1-\alpha}\|x_0 - x_1\| \to 0 \text{ as } n \to \infty.$$



2) Completeness of X $\Rightarrow$ $x_n \to x \in X$.
Continuity of T $\Rightarrow$ taking limit in $Tx_{n} = x_{n+1}$, we obtain $Tx = x$. QED
~~$Tx = T(\lim x_n) = \lim(T(x_n)) = \lim(x_{n+1}) = x$~~

~~QED~~

Remark: Thm holds for a complete metric space (linearity never used).

Example : (Fredholm integral equations of second kind)

for an unknown $f : [0,1] \to \mathbb{R}$:

$$f(t) - \int_0^1 \underbrace{K(t,s)}_{\text{"Kernel"}: [0,1]\times[0,1]\to\mathbb{R}} f(s)\,ds = g(t). \qquad (*)$$

This equation can be rewritten as a fixed point equation

$$Tx = x$$

where the map T is defined as

$$(Tf)(t) := g(t) + \int_0^1 K(t,s) f(s)\,ds.$$

We apply the Fixed Pt Thm. in $C[a,b]$.
So, Assumptions: $g \in C[0,1]$, $K \in C([0,1]\times[0,1])$

~~⇒ T is a~~

Check the contractivity:

$$\|Tf - Tg\|_\infty = \sup_{a\le t\le b} \left| \int_0^1 K(t,s) \underbrace{(f(s)-g(s))}_{\max \text{ outside}} ds \right|$$

$$\le \cdot \|K\|_\infty \cdot \|f-g\|_\infty$$

⇒ Contraction if $\|K\|_\infty < 1$.

Prop Suppose ~~$f\in C[0,1]$, $K\in C$~~ $g(t)$ and $K(t,s)$ are continuous functions, and $\|K\|_\infty < 1$. Then there exists a unique solution $f(t)$ of integral eq. $(*)$. It can be found by successive iteration from $\forall f_0(t)$

$$f_n(t) = g(t) + \int_0^1 K(t,s) f_{n-1}(s)\,ds.$$

HW Exercise: $\sqrt{a}$

Hilbert spaces.

(inner product)
Def Let E be a ~~complex~~ linear space over $\mathbb{C}$.
An inner product on E is a ~~function $\langle x,y\rangle$:~~
function $\langle \cdot,\cdot \rangle : E \to \mathbb{C}$ satisfying

(i) $\langle x,x\rangle \geq 0$ for all $x \in E$; $\langle x,x\rangle = 0$ iff $x=0$;

(ii) $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$

for every $x,y,z \in E$; $a,b \in \mathbb{C}$

(iii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ for all $x,y \in E$.

E is then called an inner product space

Remarks 1) It follows that ~~The~~ An inner product satisfies:

(iv) $\langle x, ay+bz\rangle = \bar{a}\langle x,y\rangle + \bar{b}\langle x,z\rangle$

$$\left[\underset{\overline{\langle ay+bz, x\rangle}}{\|} = \overline{a\langle y,x\rangle} + \overline{b\langle z,x\rangle} = \bar{a}\langle x,y\rangle + \bar{b}\langle x,z\rangle \right]$$

~~(iv) $\langle x, ay\rangle = a$~~

2) Over $\mathbb{R}$ — similar, except no conjugation in (iii).

3) $x \perp y \overset{\text{def}}{\iff} \langle x,y\rangle = 0$.

Examples: 1) $\mathbb{C}^n$, $\langle x,y\rangle = \sum x_i \bar{y}_i$

2) $\mathbb{R}^n$, $\langle x,y\rangle = \sum x_i y_i$

~~**Proposition** An inner product space is automatically a normed space, with the norm defined by $\|x\| = \langle x,x\rangle^{1/2}$~~

[Norm axioms

THM (Cauchy–Schwartz inequality.) Let X be an inner product space. Then for all $x, y \in X$:

$$|\langle x, y\rangle| \le \langle x, x\rangle^{1/2} \langle y, y\rangle^{1/2}.$$

Notation: $\|x\| = \langle x, x\rangle^{1/2}$. $\overset{?}{\Rightarrow}$ $\boxed{|\langle x,y\rangle| \le \|x\| \|y\|}$

1) Assume $\langle x, y\rangle \in \mathbb{R}$. For $t \in \mathbb{R}$,

$$0 \le \langle x+ty, x+ty\rangle = t^2 \|y\|^2 + 2t\langle x, y\rangle + \|x\|^2.$$

A quadratic polynomial $\underbrace{\ge 0}_{\text{everywhere}}$ iff its discriminant $\mathcal{B} \le 0$, i.e.

$$\langle x, y\rangle^2 - \|x\|^2\|y\|^2 \le 0.$$

QED.

2) General $\langle x, y\rangle \in \mathbb{C}$.

$$\langle x, y\rangle = |\langle x, y\rangle| e^{i \operatorname{Arg}\langle x, y\rangle} \Rightarrow$$

$$|\langle x, y\rangle| = \langle x, y'\rangle \quad \text{where} \quad y' = e^{i \operatorname{Arg}\langle x, y\rangle} y.$$

Applying case (1) for

$$\langle x, y'\rangle \le \|x\| \|y'\| = \|x\| \|y\|.$$

Cor An inner product space X is automatically a normed space, with the norm defined as

$$\|x\| := \langle x, x\rangle^{1/2}.$$

The inner product is continuous on $X \times X$. (by $*$)

Only the Δ inequality is non-trivial:

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2$$
$$\le \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 \quad \text{by C-S}$$
$$= (\|x\| + \|y\|)^2.$$

QED.