Example: L_2.

Recall that $L_2(\Omega, \Sigma, \mu)$ is the Banach space of functions f on Ω s.t.

$$\|f\|_2 = \left(\int |f|^2 \, d\mu \right)^{1/2} < \infty$$

Let us define the inner product on L_2 by

$$\langle f, g \rangle := \int f \bar{g} \, d\mu.$$

Note that:

1) $\langle f, g \rangle$ is indeed defined for all $f, g \in L_2$

We can first define it on the dense set of simple functions in L_2. Then extend by continuity (see Cor. p. 28 and exercise) to the whole L_2.

2) The norm $\|\cdot\|_2$ is indeed induced by the inner product $\langle \cdot, \cdot \rangle$ as $\|f\|_2 = \langle f, f \rangle^{1/2}$.

3) Cauchy-Schwartz inequality then reads

$$\boxed{\left| \int f \bar{g} \, d\mu \right| \leq \left(\int |f|^2 d\mu \right)^{1/2} \left(\int |g|^2 d\mu \right)^{1/2} \quad \forall f, g \in L_2}$$

Exercise: LHS $\to \int |fg| \, d\mu$ (apply the ineq. for $|f|, |g|$).

4) Cauchy-Schwartz inequality is a partial case of ~~the~~ Hölder's inequality: $\forall$ $p, q > 0$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$,

$$\boxed{\cancel{\left| \int f g \, d\mu \right|} \int |fg| \, d\mu \leq \left(\int |f|^p d\mu \right)^{1/p} \left(\int |g|^q d\mu \right)^{1/q} \quad \forall f \in L_p, g \in L_q}$$

(proof see e.g. in [EMT] Thm 1.2.4).

✓ Parallelogram law - see p. 179

✓ X/Y Banach

Example: ℓ_2

Recall that ℓ_2 is the Banach space of sequences $x = (x_i)_1^\infty$ s.t.

$$\|x\|_2 = \left(\sum_i |x_i|^2\right)^{1/2} < \infty.$$

Repeating the ~~analog~~ example of L_2 above (actually noting that ℓ_2 is a partial case of $L_2(\Omega, \Sigma, \mu)$ with $\Omega = \mathbb{N}$, μ = counting measure), we define the inner product on ℓ_2 by

$$\langle x, y \rangle = \sum_i x_i \bar{y}_i.$$

(as in $\mathbb{C}^n$)

- Cauchy-Schwartz inequality then reads:

$$\boxed{\left|\sum_i x_i \bar{y}_i\right| \le \left(\sum |x_i|^2\right)^{1/2} \left(\sum |y_i|^2\right)^{1/2}}$$

Ex. $\longrightarrow$ LHS $\to \sum_i |x_i y_i|$.

- More generally,

Hölder's inequality $\forall\, p, q > 0$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum_i |x_i y_i| \le \left(\sum_i |x_i|^p\right)^{1/p} \left(\sum_i |y_i|^q\right)^{1/q} \qquad \forall\, x \in L_p,\ y \in L_q.$$

Example $M_{m,n}$:= all $m \times n$ matrices with complex entries.

$\langle A, B \rangle := tr(A^* B)$ where A^* = Hermitian conjugate of A.

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{a}_{ij} b_{ij}$$

This is clearly an inner product; ~~can be the same~~ can be obtained by identification of $M_{m,n}$ with $\mathbb{C}^{mn}$. The corresponding norm

$$\|A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

is called Hilbert-Schmidt (also Frobenius norm).

Def A complete inner product space is called <u>Hilbert space</u>.

<u>Examples</u>: $L_2(\Omega, \Sigma, \mu)$, ℓ_2, ℓ_2^n, $\mathbb{C}$, $M_{m,n}$ $(\equiv \ell_2^{mn})$

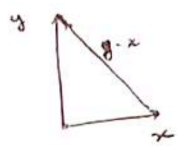
<u>Example</u>. $L_2[a,b]$ can be defined as a completion of $C[a,b]$ w.r.to $\|\cdot\|_2$-norm.

~~Examples the space of all m×n matrices with complex entries $M_{m,n}$ $\langle A, B\rangle = tr(A$~~

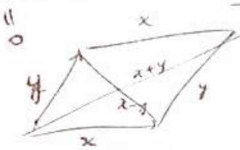
~~Geometry of Orthogonality in Hilbert space.~~

Orthogonality, Projections

Prop (<u>Pythagorean thm</u>) If $x \perp y$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$



$$\left[\ \|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \underbrace{\langle x,y\rangle + \langle y,x\rangle}_{=0} + \|y\|^2 \ \right]$$



<u>Prop</u> (Parallelogram law) $\boxed{\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2}$

Def (<u>orthogonal complement</u>) For a subset $A \subset X$, $A^\perp := \{x \in X : x \perp a \text{ for all } a \in A\}$

~~In other words Equivalently~~

Prop $\boxed{A^\perp \text{ is a closed linear subspace}}$

~~$A^\perp = \bigcap_{a \in A} a^\perp \Rightarrow$ it suffices to show that for $a \in X$, $a^\perp$ is closed~~

1) Linear subspace — straightforward.

2) Closedness: $A^\perp = \bigcap_{a \in A} a^\perp$. So it suffices to show that $\forall a \in X$, $a^\perp$ is a closed set.

Indeed, if $x_n \in a^\perp$ and $x_n \to x \in X$ then $0 = \langle x_n, a\rangle \to \langle x, a\rangle \Rightarrow x \in A^\perp$ QED]

Prop

Prop. $\boxed{A^\perp \cap A = \{0\}}$ $\ulcorner$ If $x \in A^\perp \cap A$, then $\langle \underset{A}{\overset{\cap}{x}}, \underset{A^\perp}{\overset{\cap}{x}} \rangle = 0 \Rightarrow x = 0$ $\lrcorner$.

Example: Consider the subspace ~~of L_2~~: Y of constant functions in L_2

~~$Y := \{ f \in L_2 : \int f \, d\mu = 0 \}$.~~

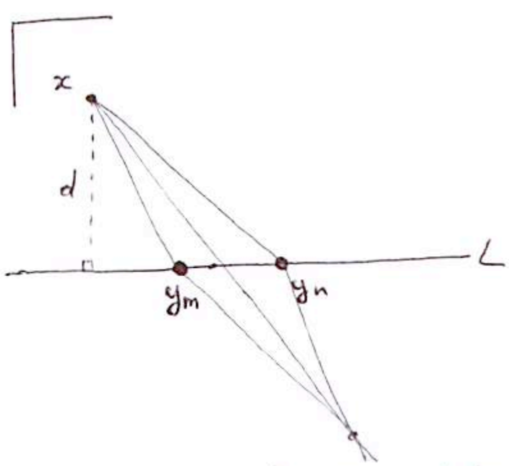
Then $Y^\perp = \{ f \in L_2 : \int f \, d\mu = 0 \}$

THM (Projection). Let Y be a closed subspace of a Hilbert space X.

(i) For every $x \in X$, there exists a unique closest point $y \in Y$, i.e.

$$\|x - y\| = \min_{y' \in Y} \|x - y'\|.$$

(ii) The point $y \in Y$ closest to $x \in X$ is the unique vector in Y such that $x - y \in Y^\perp$.



(i) Existence. Denote the distance $d := \min_{y' \in Y} \|x - y'\|$.

Choose a sequence (y_n) in Y which satisfies

$$\|x - y_n\| \to d.$$

It suffices to show that (y_n) is Cauchy, for then $y_n \to y \in Y$ by completeness of Y. By continuity of the norm it would also follow that $\|x - y\| = d$.

To bound $\|y_n - y_m\|$ we use Parallelogram Law:

$$\|y_n - y_m\|^2 + \underbrace{4\|x - \tfrac{1}{2}(y_n - y_m)\|^2}_{\geq 4d^2} = \underbrace{2\underbrace{\|x - y_n\|^2}_{\to d^2} + 2\underbrace{\|x - y_m\|^2}_{\to d^2}}_{4d^2}.$$

Hence $\limsup \|y_n - y_m\|^2 = 0$. QED.

Uniqueness: If there were two different closest points y', y'', then the alternating sequence $y_{2n} := y'$, $y_{2n+1} := y''$ would not be Cauchy. ↯

Remark: Part (i) holds $\forall$ closed convex set Y in X (not necessarily a linear subspace), and with the same proof.