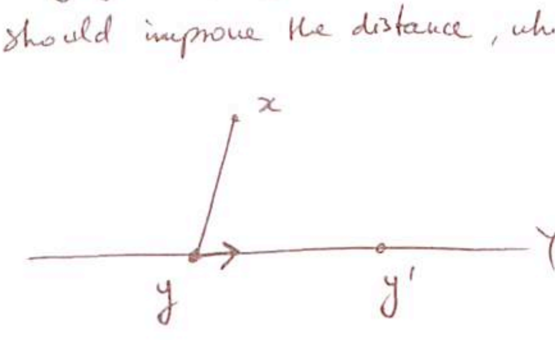
LECTURE 8

(ii) Assume that $x-y \notin Y^{\perp}$, i.e. $\exists y' \in Y : \langle x-y, y' \rangle \neq 0$.

~~$\exists y' \in Y : \langle y\ x-y'$~~

~~Multiplying y'?~~

$\Rightarrow$ Sliding y slightly in the direction of y' should improve the distance, which is impossible:



For $t \in \mathbb{C}$,

~~$\|y-x\|^2 \le \|y$~~

$$\|x-y\|^2 \le \|x-y+ty'\|^2 = \|x-y\|^2 + 2\,\mathrm{Re}\,t\langle x-y, y'\rangle + t^2\|y'\|^2$$

$$\Rightarrow \quad t^2\|y'\|^2 + 2\,\mathrm{Re}\left(t\,\langle x-y, y'\rangle\right) \ge 0 \quad \forall\, t \in \mathbb{C}$$

This is impossible for small t ~~because~~

(take $t := \varepsilon\, \overline{\langle x-y, y'\rangle}$ and let $\varepsilon \to 0$).

<u>Uniqueness</u>: Suppose $x-y' \in Y^{\perp}$, $x-y'' \in Y^{\perp}$.

~~Adding it~~ Subtracting yields $y'-y'' \in Y^{\perp}$.

But $y'-y'' \in Y \Rightarrow y'-y'' = 0$ ~~by Prop~~. since $Y^{\perp} \cap Y = \{0\}$.

<u>Cor</u> Let Y be a closed subspace of a Hilbert space X. Then $\forall\, x \in X$ can be uniquely represented as

$$x = y + z, \quad y \in Y, \quad z \in Y^{\perp}.$$

~~P_y is called the ortho~~

~~Def (Orthogonal direct sum) let Y, Z be closed subspaces of a Hilbert space X. The orthogonal direct sum is defined as $Y \oplus Z = \{y$~~

This is sometimes written as $\boxed{X = Y \oplus Y^{\perp}}$ (orthogonal decomposition ~~direct sum~~)

Exercise $(Y^{\perp})^{\perp} = Y$.

This is sometimes written as $\boxed{X = Y \oplus Y^{\perp}}$ (orthogonal decomposition)

y is called the orthogonal projection of x onto Y; denoted

$$\boxed{y = P_Y x}$$

Orthogonal ~~series~~ systems

X: Hilbert sp.

Def An orthogonal system is a sequence of vectors (x_k) in X s.t.

$$\langle x_k, x_\ell \rangle = 0 \quad \text{for all } k \neq \ell.$$

If, additionally $\|x_k\| = 1$ for all k, the system is called orthonormal.

~~$\langle x_k, x_k \rangle$~~

Equivalently, (x_k) is orthonormal if $\langle x_k, x_\ell \rangle = \delta_{k\ell}$ $\forall k, \ell$.

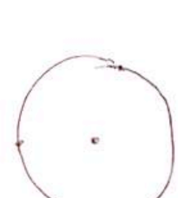
Example (canonical basis in ℓ_2).

$$x_k := (0, \dots, 0, \underset{\substack{\uparrow \\ k\text{'th coordinate}}}{1}, 0, \dots)$$

$(x_k)_1^\infty$ is clearly orthonormal.

Example (Fourier basis in L_2).

$L_2[-\pi, \pi]$ (sometimes denoted $L_2(\mathbb{T})$) also $L_2[0, 2\pi]$;



$$f_k(t) := \frac{1}{\sqrt{2\pi}} e^{ikt}, \qquad t \in [-\pi, \pi]$$

Prop $\boxed{(f_k)_{-\infty}^{\infty} \text{ is an orthonormal system in } L_2[-\pi, \pi].}$

Pf Straightforward: $\langle f_k, f_\ell \rangle = \int_{-\pi}^{\pi} f_k(t) \overline{f_\ell(t)}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-\ell)t}\, dt = \delta_{k\ell}$

A closely related variant of this system ~~can be oft~~ are: trigonometric ~~monomials~~ system: note that

$$\cancel{\mathrm{Re}\, f_k(t) = \frac{\xi}{\sqrt{2\pi}}}$$

$$f_k(t) = \frac{1}{\sqrt{2\pi}} \left(\cos(kt) + i \sin(kt) \right)$$

Considering then the real and complex parts separately, and doing a similar straightforward verification, we obtain:

Prop The trig. system $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \cos 2t, \frac{1}{\sqrt{\pi}} \sin 2t, \dots \right\}$ is orthogonal in $L_2[-\pi, \pi]$.

Now, for a theory of orthogonal series.

Lemma (Pythagorean thm) Every orthogonal system satisfies

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2 \qquad \forall k.$$

[~~Ex: either by induction from n=2, or directly~~

$$\left\langle \sum_{k=1}^{n} x_k, \sum_{l=1}^{n} x_l \right\rangle = \sum_{k,l=1}^{n} \langle x_k, x_l \rangle = \sum_{k=1}^{n} \langle x_k, x_k \rangle^2 \text{]}$$

THM (Orthogonal series) Let (x_k) be an orthogonal system in H. The series $\sum_k x_k$ converges iff $\sum_k \|x_k\|^2 < \infty$.

~~Remarks 1)~~

By Cauchy criterion, $\sum_k x_k$ converges $\Leftrightarrow \left\| \sum_{k=n}^{m} x_k \right\|^2 \to 0$, $n, m \to \infty$

$$\left\| \sum_{k=n}^{m} x_k \right\|^2 = \sum_{k=n}^{m} \|x_k\|^2 \quad \text{by lemma}$$

~~which is~~ Again, by Cauchy criterion, this is equivalent to $\sum_k \|x_k\|^2 < \infty$

~~Remark~~

~~An orthogonal series~~

~~Cor $\sum_k x_k$ converges $\Rightarrow$ converges unconditionally, i.e. for $\forall$ reordering of terms~~

~~The series $\sum_k \|x_k\|^2$ converges absolutely $\Rightarrow$ in $\forall$ order.~~

Remark

<u>Remarks</u> 1) If $\sum_k x_k$ converges then $\left\| \sum_k x_k \right\|^2 = \sum_k \|x_k\|^2$ (by taking the limit of partial sums)

2) If $\sum x_k$ converges then it converges <u>unconditionally</u>, i.e. for $\forall$ reordering of terms.

($\sum_k \|x_k\|^2$ converges absolutely in $\mathbb{R}$ $\Rightarrow$ unconditionally)

Fourier series

Let (x_n) be an orthonormal system in X.

The Fourier series ("orthogonal expansion") of a vector $x \in X$ is the formal series

$$\sum_k \langle x, x_k \rangle x_k .$$

The coefficients $\langle x, x_k \rangle$ are called Fourier coeff's of x.

Example – {recall ~~trigonometric or~~ expon. system in L_2}:

$$\langle f_k, f \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt =: \hat{f}(k)$$

Fourier series is $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt}$ $\quad$ ($= f(t)$ as will show).

THM (Bessel's Inequality) For an orthonormal system (x_n) and for every vector $x \in X$ one has

$$\sum_k |\langle x, x_k \rangle|^2 \le \|x\|^2 .$$

Pf Consider partial sum of Fourier series; ~~we claim that~~ and write

$$x = \underbrace{\left(x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k \right)} + \underbrace{\sum_{k=1}^{n} \langle x, x_k \rangle x_k}.$$

~~We claim that these two vectors are orthogonal.~~

We claim that the vectors

$$x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k , \quad x_1, x_2, \dots, x_n$$

are orthogonal. This is a simple check, e.g.

$$\left\langle x - \sum_{k=1}^{n} \langle x, x_k \rangle x_k , x_1 \right\rangle = \langle x, x_1 \rangle - \langle x, x_1 \rangle \underbrace{\langle x_1, x_1 \rangle}_{=1} = 0$$