~~Hence~~ **THM** (Bessel's inequality) Let (x_k) be an orthonormal system. Then
$$\sum_k |\langle x, x_k\rangle|^2 \le \|x\|^2 \quad \text{for all } x \in X.$$

Consider a partial sum of Fourier series
$$S_n = \sum_{k=1}^n \langle x, x_k\rangle x_k,$$
and use Cauchy-Schwartz inequality:
$$\underbrace{|\langle S_n, x\rangle|}_{\substack{\| \\ \sum_{k=1}^n |\langle x, x_k\rangle|^2 \\ \text{by def. of } S}} \le \underbrace{\|S_n\|}_{\substack{\| \\ \left(\sum_{k=1}^n |\langle x, x_k\rangle|^2\right)^{1/2} \\ \text{by Pythagorean thm}}} \cdot \|x\|$$

$$\Rightarrow \sum_{k=1}^n |\langle x, x_k\rangle|^2 \le \|x\|^2.$$
Letting $n \to \infty$ completes the proof.

Corollary The Fourier series of every vector x converges in X.

By the convergence criterion for orthogonal series (Thm p.34), it suffices to check that
$$\sum_k \|\langle x, x_k\rangle x_k\|^2 < \infty$$
But this $= \sum_k |\langle x, x_k\rangle|^2$ which converges by Bessel's inequality.

~~Prop~~ (~~[illegible]~~ optimality)

Simple geometric property of Fourier series: Recall $S_n = \sum_1^n \langle x, x_k \rangle x_k$.

Define span

for finite sets, $\mathrm{span}(x_1 \dots x_n) = \{ \sum_{k=1}^n a_k x_k : a_k \in \mathbb{C} \}$

For infinite, $\mathrm{span}(x_k) = \{ \sum_1^n a_k x_k : n \in \mathbb{N}, a_k \in \mathbb{C} \}$

<u>Lemma</u> $\boxed{x - S_n \perp x_k \text{ for all } k = 1 \dots n}$

$$\langle x - S_n, x_k \rangle = \langle x, x_k \rangle - \underbrace{\langle S_n, x_k \rangle}_{= \langle x, x_k \rangle \text{ by def. of } S} = 0.$$

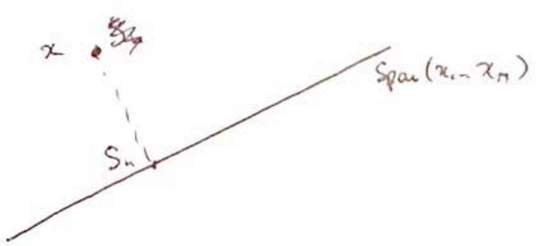
Hence $x - S_n \perp \mathrm{Span}(x_1, \dots, x_n)$.

$\Rightarrow$ writing $x = S_n + (x - S_n)$ we see that ~~the~~ two terms here are orthogonal $\Rightarrow$

<u>Cor</u> The partial sum S_n of Fourier series of x is the orthogonal projection of x onto $\mathrm{Span}(x_1 \dots x_n)$.

$\Rightarrow$

By the Theorem on projections (p. 32), S_n is the closest point to x in $\mathrm{span}(x_1 \dots x_n)$



$\Downarrow$

<u>Cor</u> (Optimality). Among all linear combinations $S_n' = \sum_{k=1}^n b_k x_k$, the partial sum of Fourier series $S_n = \sum_{k=1}^n \langle x, x_k \rangle x_k$ minimizes the approx error $\|x - S_n'\|$.

~~Similar statement Fact~~ Letting $n \to \infty$ in the arguments above, $\Rightarrow$

<u>Thm</u> (Optimality) 1) The Fourier series of x w.r.to (x_k) is the orthog. projection of x onto $\overline{\mathrm{Span}}(x_k)$.

2) Among all series $S' = \sum a_k x_k$, the Fourier series of x minimizes the approx. error $\|x - S'\|$.

(Exercise)

Orthonormal bases, Parseval Identity

Def A system (x_k) in ~~Banach normed space is~~ a ~~[illegible]~~ Banach space X is complete if $\overline{\text{span}}\,(x_k) = X$.
A complete orthonormal system in a Hilbert space X is called an ~~orthogonal~~ orthonormal basis.

~~[illegible]~~ (Fourier expansion)

THM Let (x_k) be an orthonormal basis. Then every $x \in X$ ~~[illegible]~~ can be expanded in Fourier series:

$$x = \sum_k \langle x, x_k \rangle x_k .$$

~~Moreover~~ Consequently

$$\|x\|^2 = \sum_k |\langle x, x_k \rangle|^2 \qquad \text{(Parseval's Identity)}$$

The first part follows from the Optimality Thm (p 39) since, by completeness, $\overline{\text{span}}(x_k) = X$, so the projection is the identity map on X

The second part follows from Pythagorean thm (see Thm on Orthogonal series p.35 and the remark below it).

Remark ~~Bessel's inequality~~ Parseval's identity is an equality case in Bessel's inequality; it holds if and only if the ~~sys~~ orthonormal system is complete [EMT Thm 2.1.12]

Examples 1) By Weierstrass theorem, the monomials $\{t^k\}_{k\geq 0}$ ~~is a~~ form a complete system in $C[0,1]$
We claim that $\{t^k\}_{k\geq 0}$ ~~[illegible]~~ is also a complete system in $L_2[0,1]$

Indeed, $C[0,1]$ is dense in $L_2[0,1]$ as we noted before.
Thus, $\forall f \in L_2[0,1]$, $\forall \varepsilon > 0$ $\exists g \in C[0,1]$. $\|f-g\|_2 < \varepsilon$.
For $g \in C[0,1]$, $\exists h \in \overline{\text{span}}\{t^k\}$ s.t. $\|g-h\|_\infty \leq \varepsilon$
$$\underset{\|g-h\|_2}{\overset{\ }{\geq}}$$
$\Rightarrow \|f-h\|_2 \leq 2\varepsilon$.

2) The trigonometric system $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \dots \right\}$
is complete in ~~$C[0,1]$ (this is~~
the space of continuous 2π-periodic functions on $[-\pi, \pi]$.
(this is a version of Weierstrass thm).
Consequently, by the same argument as in (1),
the trigonometric system is complete in $L_2(-\pi, \pi)$ $\Rightarrow$ orthonormal basis.

3) By a similar argument, the exponential system $(e^{ikt})_{k \in \mathbb{Z}}$
is an orthonormal basis in $L_2(-\pi, \pi)$.

Writing (3) in the functional form:

Fourier Series

$\forall f \in L_2(-\pi,\pi)$, can be expanded in $L_2(-\pi,\pi)$ as

$$f = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{ikt}$$

where $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$

~~The convergence~~ is