Final Exam

Math 425-201 Su10

by Prof. Michael "Cap" Khoury

Name:	

Directions:

Please print your name legibly in the box above.

You have 110 minutes to complete this exam. You may use any type of conventional calculator, but no computers/cell phones/iPads/etc. You may use three 3×5 notecards (front and back) but no other reference materials of any kind. There are ten problems which are worth 20 points each. Partial credit is possible. *For any opportunity for partial credit, clear and relevant work must be shown.*

You choose *nine* of the ten problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, then whichever problem is skipped by most students becomes the default skip. This is not what you want.

This exam is worth 180 points, representing 45% of the 400 total points possible in the course.

This exam consists of 6 pages, front and back (one side for this cover page, one side for a table of values of Φ, and one side for each of 10 problems). If you do not have all 6 pages or if any sides are blank, please notify me immediately!

Problem Number	Value	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
TOTAL	**180**	

Problem 1

SKIP THIS PROBLEM ☐

You all know it's coming, so let's get it over with.

The $N = 50000$ variables $X_1, X_2, X_3, \ldots, X_N$ are independent, identically distributed random variables supported on $[0,1]$, each having the density $f(x) = 6x(1-x)$.

Remark: the X_i have a beta distribution, so you may have a formula for parts (a) and (b) in your "inventory".

a) [5 pts] What is $E[X_{19}]$?

$$E[X_{19}] = \int_0^1 6(x^2 - x^3)\,\mathrm{d}x = \left[2x^3 - (3/2)x^4\right]_0^1 = 1/2$$

b) [5 pts] What is $\mathrm{Var}[X_{19}]$?

$$E[X_{19}^2] = \int_0^1 6(x^3 - x^4)\,\mathrm{d}x = \left[(3/2)x^4 - (6/5)x^5\right]_0^1 = 3/10$$

$$\mathrm{Var}[X_{19}] = E[X_{19}^2] - E[X_{19}]^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}$$

c) [10 pts] Let $Y = \frac{X_1 + X_2 + \cdots + X_N}{N}$ be the average of the X_i. Estimate the probability $P(0.501 \leqslant Y \leqslant 0.502)$. (Assume that N is large enough that the Central Limit Theorem gives an adequate estimate.)

First, Y is approximately normal with $\mu = 0.5$ and $\sigma = \frac{\sqrt{1/20}}{\sqrt{N}} = 0.001$; let Z be a standard normal variable.

$$P(0.501 \leqslant Y \leqslant 0.502) \approx P(1 \leqslant Z \leqslant 2) = \Phi(2) - \Phi(1) \approx 0.1359$$

Problem 2

SKIP THIS PROBLEM □

Die-Rollin' Dave has a set of ten fair 6-sided dice. Each of these dice has 3 red faces, 2 blue faces, and 1 green face. He rolls them all.

a) [5 pts] What is the probability that 4 dice show red, 3 show blue, and 3 show green?

The numbers of red, blue, and green faces are multinomially distributed.

$$\frac{10!}{4!3!3!}\left(\frac{1}{2}\right)^4\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^3$$

b) [7 pts] What is the probability that all the dice turn up the same color?

$$\left(\frac{1}{2}\right)^{10}+\left(\frac{1}{3}\right)^{10}+\left(\frac{1}{6}\right)^{10}$$

c) [8 pts] Dave's friend Katie has a strange form of color-blindness; she can tell whether or not two objects are the same color, but cannot identify the color. If she rolls the ten dice and sees that four of the dice are the same color, three of the dice are a second color, and the other three are the third color, what is the probability that there are four dice showing red faces?

Let A_R be the event that there are 4 red faces, 3 blue, 3 green; let A_B be the event that there are 3 red faces, 4 blue, 3 green; let A_G be the event that there are 3 red faces, 3 blue, 4 green.

This problem is asking for $P(A_R|A_R \sqcup A_B \sqcup A_G)$.

$$\begin{aligned}\frac{P(A_R)}{P(A_R)+P(A_B)+P(A_G)} &= \frac{\frac{10!}{4!3!3!}\left(\frac{1}{2}\right)^4\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^3}{\frac{10!}{4!3!3!}\left[\left(\frac{1}{2}\right)^4\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^3+\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^4\left(\frac{1}{6}\right)^3+\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^4\right]} \\ &= \frac{1/2}{1/2+1/3+1/6} \\ &= 1/2\end{aligned}$$

Problem 3

SKIP THIS PROBLEM □

Datura and Gabriel are playing a game with a fair 8-sided die (numbered 1 through 8). They repeatedly roll the die. Datura is seven years old; her score is the number of rolls required to turn up a 7. Gabriel is one year old; his score is the number of rolls required to turn up a 1. Let D be Datura's score; let G be Gabriel's score.

(For example, if the sequence of rolls begins $7, 3, 4, 7, 3, 1, 8, 5, \ldots$, then $D = 1, G = 6$.)

a) [5 pts] What is $E[D]$?

D is geometric with $p = 1/8$. $E[D] = 8$.

b) [5 pts] Compute $E[D|G = 1]$.

If $G = 1$, then certainly $D > 1$, and moreover $D - 1$ is geometric with $p = 1/8$. Thus $E[D|G = 1] = 9$.

c) [6 pts] Compute the conditional probability $P(D = 1|G = 2)$.

If $G = 2$, then the first die roll was *not* a 1. The probability of rolling a 7, given that we did not roll a 1, is $1/7$. $P(D = 1|G = 2) = 1/7$.

d) [4 pts] Compute the probability $P(D > G)$.

Trick question. $1/2$

Problem 4

SKIP THIS PROBLEM ☐

Let X be a exponentially distributed random variable with rate 1, let Y be a uniform random variable on the interval $[0,1]$, and suppose that X, Y are independent.

Let $Z = X + Y$.

a) [5 pts] Compute $E[Z]$.

$$E[Z] = E[X] + E[Y] = 1 + (1/2) = 3/2$$

b) [15 pts] Compute the density function for Z, $f_Z(z)$.

Hint: There is one formula for $0 < z < 1$ and another for $z > 1$.

We obtain the density of Z as a convolution of the densities of X and Y.

For $0 < z < 1$, we have the following.

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x) \mathrm{d}x = \int_0^z e^{-x} \mathrm{d}x = 1 - e^{-z}$$

For $z > 1$, we have instead the following.

$$f_Z(z) = \int_{z-1}^z f_X(x) f_Y(z-x) \mathrm{d}x = \int_{z-1}^z e^{-x} \mathrm{d}x = e^{-(z-1)} - e^{-z} = (e-1)e^{-z}$$

Problem 5

SKIP THIS PROBLEM □

Suppose that Professor Vector has three computers, all three of which are still working. The functional lives of the computers (in years) are independent exponential random variables (each with a different parameter λ, as indicated below).

- Her netbook is brand new. The average lifetime of this type of netbook is 2 years.
- Her laptop is 1 year old. The average lifetime of this type of laptop is 3 years.
- Her trusty desktop computer is 8 years old. The average lifetime of this type of computer is 5 years.

Let X, Y, Z be the additional lifetime of the netbook, laptop, and desktop, respectively. By memorylessness of the exponential, it is irrelevant how long Professor Vector has already had each computer. Then $f_X(x) = \frac{1}{2}e^{-1/2}$, $f_Y(y) = \frac{1}{3}e^{-1/3}$, $f_Z(z) = \frac{1}{5}e^{-1/5}$.

a) [6 pts] What is the probability that all three of Professor Vector's computers will still be functional in 1 year?

$$P(X > 1, Y > 1, Z > 1) = P(X > 1)P(Y > 1)P(Z > 1) = e^{-1/2}e^{-1/3}e^{-1/5} = e^{-31/30}$$

b) [6 pts] What is the probability that at least one of Professor Vector's computers will still be functional in 1 year?

$$\begin{aligned} P(X > 1 \cup Y > 1 \cup Z > 1) &= 1 - P(X < 1)P(Y < 1)P(Z < 1) \\ &= 1 - (1 - e^{-1/2})(1 - e^{-1/3})(1 - e^{-1/5}) \end{aligned}$$

c) [8 pts] What is the probability that her laptop will still be working when her desktop computer fails?

$$P(Y > Z) = \int_0^\infty \int_z^\infty \frac{1}{15} e^{-y/3} e^{-z/5}\, \mathrm{d}y\, \mathrm{d}z = \int_0^\infty \frac{1}{5} e^{-z/3} e^{-z/5}\, \mathrm{d}y\, \mathrm{d}z = \frac{3}{8}$$

Problem 6

SKIP THIS PROBLEM ☐

Let X be a continuous positive-valued random variable with the following distribution.

$$F_X(x) = \begin{cases} \frac{x}{x+1} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

a) [5 pts] Compute the density function of X, $f_X(x)$.

$$f_X(x) = F_X'(x) = \begin{cases} \frac{1}{(x+1)^2} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

b) [4 pts] Compute $P(X < 2)$.

$$P(X < 2) = F(2) = \frac{2}{3}$$

c) [5 pts] Compute $P(X > 1 | X < 2)$.

$$P(X > 1|X < 2) = \frac{P(1 < X < 2)}{P(X < 2)} = \frac{F(2) - F(1)}{F(2)} = \frac{2/3 - 1/2}{2/3} = \frac{1}{4}$$

d) [6 pts] Let $Y = X^2$. Compute the density function of Y, $f_Y(y)$.

Note that Y is supported on $(0, \infty)$, so $f_Y(y) = 0$ for negative y. For $y \geqslant 0$, we have the following.

$$\begin{aligned} F_Y(y) = P(Y \leqslant y) &= P(X^2 \leqslant y) \\ &= P(X \leqslant \sqrt{y}) \\ &= F_X(\sqrt{y}) \\ &= \frac{\sqrt{y}}{\sqrt{y}+1} \\ f_Y(y) = F_Y'(y) &= \frac{1}{2\sqrt{y}(\sqrt{y}+1)^2} \end{aligned}$$

Problem 7

SKIP THIS PROBLEM ☐

Alice and Betsy play 900 rounds of rock-paper-scissors (all you need to know about rock-paper-scissors for this problem is that it is a two-player game in which ties are possible). Let us assume that neither player has any advantage, so that in each round, independently of all other rounds, the outcomes "Alice wins", "Betsy wins", and "tie" are equally likely. Let X be the number of rounds Alice wins, and let Y be the number of rounds that were not ties.

X is binomial with $n = 900, p = 1/3$; Y is binomial with $n = 900, p = 2/3$.

a) [5 pts] What is $E[X]$?

300

b) [5 pts] What is $\mathrm{Var}[X]$?

200

c) [5 pts] What is $E[Y]$?

600

d) [5 pts] What is $\mathrm{Var}[Y]$?

200

Problem 8

SKIP THIS PROBLEM ☐

The random variables X, Y are jointly continuously random variables with the following joint density, where k is a constant.

$$f_{XY}(x,y) = \begin{cases} k(xy+1) & 0 \leqslant x \leqslant 1, 1 \leqslant y \leqslant 2 \\ 0 & \text{else} \end{cases}$$

a) [5 pts] What is k?

$$\int_0^1 \int_1^2 (xy+1)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left(\frac{3}{2}x+1\right)\mathrm{d}x = \frac{7}{4}$$

Since the density must integrate to 1 over the support, we must have $k = \frac{4}{7}$.

b) [5 pts] Compute the marginal density $f_Y(y)$, for $1 \leqslant y \leqslant 2$.

$$f_Y(y) = \int_0^1 \frac{4}{7}(xy+1)\,\mathrm{d}x = \frac{2}{7}y + \frac{4}{7}$$

c) [5 pts] Compute the conditional density $f_{X|Y}(x|y)$ for $0 \leqslant x \leqslant 1, 1 \leqslant y \leqslant 2$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{4}{7}(xy+1)}{\frac{2}{7}y + \frac{4}{7}} = \frac{2xy+2}{y+2}$$

d) [5 pts] Compute $P(X < 2/3 | Y = 3/2)$.

$$F_{X|Y}(2/3|3/2) = \int_0^{2/3} \frac{2x(3/2)+2}{(3/2)+2}\,\mathrm{d}x = \frac{2}{7}\int_0^{2/3}(3x+2)\,\mathrm{d}x = \frac{4}{7}$$

Problem 9

SKIP THIS PROBLEM ☐

It's another drunken professor scenario.

Professor Whiskey has a class of n students, and he has to hand back a quiz and a homework assignment. True to form, he hands back the $2n$ papers haphazardly, uniformly randomly among all the possible ways to give each student two items. (Note that he does not necessarily give each student a quiz and a homework; some people might get two quizzes, others two homeworks.) Let X be the number of students who end up with their own quiz and homework.

a) [10 pts] Compute $E[X]$.

Let E_i be the event that student i gets both of her own items back, and let X_i be the indicator of E_i.

There are $\binom{2n}{2}$ possible pairs of papers student i might get back, one of which consists of her quiz and her homework, so $P(E_i) = \frac{1}{\binom{2n}{2}}$.

Then $X = \sum X_i$, so $E[X] = \sum_{i=1}^{n} P(E_i) = n\left(\frac{1}{\binom{2n}{2}}\right) = \frac{1}{2n-1}$.

b) [10 pts] Compute $\mathrm{Var}[X]$.

If student i gets her quiz and homework back, then some other student j will get one of $\binom{2n-2}{2}$ possible pairs of papers, so that $P(E_j|E_i) = \frac{1}{\binom{2n-2}{2}}$.

Thus $P(E_iE_j) = P(E_i)P(E_j|E_i) = \frac{1}{\binom{2n}{2}\binom{2n-2}{2}}$

Then $E\left[\binom{X}{2}\right] = \sum_{1\leqslant i<j\leqslant n} P(E_iE_j) = \binom{n}{2}\left(\frac{1}{\binom{2n}{2}\binom{2n-2}{2}}\right) = \frac{1}{2(2n-1)(2n-3)}$.

$$\mathrm{Var}[X] = 2E\left[\binom{X}{2}\right] + E[X] - E[X]^2 = \frac{1}{(2n-1)(2n-3)} + \frac{1}{2n-1} - \frac{1}{(2n-1)^2}$$

Problem 10

SKIP THIS PROBLEM □

Suppose that Alice, Betsy, and Claire all agree to meet for coffee. Each woman independently arrives at a time uniformly distributed between 2pm and 3pm. Let $X_{(1)}$ be the time (in minutes after 2pm) when the first woman arrives; let $X_{(2)}$ be the time (in minutes after 2pm) when the second arrives; and let $X_{(3)}$ be the time (in minutes after 2pm) when the third arrives.

a) [8 pts] What is the probability that Alice arrives at time $X_{(1)}$?

Alice will be the first to arrive with probability 1/3. (Here I am assuming that all three women arrive at different times, which is true with probability 1.)

b) [12 pts] Compute $P(X_{(1)} > 15, X_{(3)} < 45)$.

This is just the probability that all three women arrive between 2:15 and 2:45. This is true of each individual woman with probability 1/2, hence it is true for all three women with probability $(1/2)^3 = 1/8$.