# Final Exam (Sample Version #1)

## Math 425-001 Sp10

by Prof. Michael "Cap" Khoury

| Name: | |
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## Directions:

Please print your name legibly in the box above.

You have 110 minutes to complete this exam. You may use any type of calculator without a QWERTY keyboard, but no computers/cell phones/iPads/etc. You may use three $3 \times 5$ note-cards (front and back) but no other reference materials of any kind. There are ten problems which are worth 20 points each. Partial credit is possible. *For full credit, clear and relevant work must be shown.*

You choose *nine* of the ten problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, then whichever problem is skipped by most students becomes the default skip. This is not what you want.

This exam is worth 180 points, representing 45% of the 400 total points possible in the course.

This exam consists of 6 pages, front and back (one side for this cover page, one side for a table of $\Phi$, and one side for each of 10 problems). If you do not have all 6 pages or if any sides are blank, please notify me immediately!

| **Problem Number** | **Value** | **Score** |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| 6 | 20 | |
| 7 | 20 | |
| 8 | 20 | |
| 9 | 20 | |
| 10 | 20 | |
| **TOTAL** | **160** | |

## Problem 1

SKIP THIS PROBLEM □

A group of $n$ men and $n$ women are seated around a circular table with $2n$ seats, with each possible arrangement equally likely. Let $X$ be the number of adjacent man-woman pairs. (For example, if the people were arranged M, W, W, M, W, M, then $X$ would be 4. Note that the same person could be part of 2 pairs (or 1 pair or none).

a) [8 pts] Compute $E[X]$.

Let $X_i$ be the indicator of $E_i$, the event that positions $i-1$ and $i$ include a man and a woman. (Here we view positions 0 and $2n$ as the same place.) Then $X = \sum_{i=1}^{2n} X_i$.

Of the $\frac{2n(2n-1)}{2}$ pairs of people that might be in positions $i-1$ and $i$, $n^2$ of them are man-woman, so $P(X_i) = \frac{n^2}{n(2n-1)} = \frac{n}{2n-1}$.

$$E[X] = \sum_{i=1}^{2n} P(E_i) = 2n\left(\frac{n}{2n-1}\right) = \frac{2n^2}{2n-1}$$

b) [7 pts] Compute $E\left[\binom{X}{2}\right]$. (Careful!)

In the $2n$ cases where the pairs overlap, $P(X_iX_j) = \frac{2n(n)(n-1)}{2n(2n-1)(2n-2)} = \frac{n}{4n-2}$. In the other $n(2n-3)$ cases, $P(X_iX_j) = \frac{n^2(n-1)^2}{\binom{2n}{2}\binom{2n-2}{2}} = \frac{n(n-1)}{(2n-1)(2n-3)}$.

$$E\left[\binom{X}{2}\right] = 2n\left(\frac{n}{4n-2}\right) + n(2n-3)\frac{n(n-1)}{(2n-1)(2n-3)} = \frac{n^2}{2n-1} + \frac{n^2(n-1)}{(2n-1)} = \frac{2n^3}{(2n-1)}$$

c) [5 pts] Compute $\mathrm{Var}[X]$.

We just computed $\frac{1}{2}(E[X^2] - E[X]) = \frac{2n^3}{(2n-1)}$, so $E[X^2] = \frac{2(2n^3+n^2)}{(2n-1)}$. Finally, we compute $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \frac{2(2n^3+n^2)}{2(2n-1)} - \frac{4n^4}{(2n-1)^2} = \frac{2n^2(2n^2-1)}{(2n-1)^2}$.

## Problem 2

SKIP THIS PROBLEM □

Clark likes to buy a certain kind of lottery ticket. Each ticket, independently of all other tickets, yields one of the following prizes.

- \$10, with probability $1/20$
- \$1, with probability $1/5$
- 2 free lottery tickets, with probability $1/4$
- nothing, with probability $1/2$

What is the expected payoff of a ticket (in dollars)? (You may assume without proof that the expected payoff is finite.)

Let $X$ be the value of a ticket.

$$E[X] = 10(1/20) + 1(1/5) + 2E[X](1/4) + 0(1/2) = \frac{7}{10} + \frac{1}{2}E[X]$$

Since $E[X]$ is finite, this gives $E[X] = \$1.40$.

## Problem 3

SKIP THIS PROBLEM □

The random variables $X, Y$ are jointly distributed with $f(x,y) = \begin{cases} \frac{1}{2}(x^2 y + x y^2) & 0 \leqslant x \leqslant 2; 0 \leqslant y \leqslant 1 \\ 0 & \text{else} \end{cases}$.

a) [5 pts.] Compute the joint distribution function $F(x,y)$.

For $0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1$, we can integrate to obtain $F(x,y) = \frac{1}{12}(x^3 y^2 + x^2 y^3)$.

$$F(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ (1/12)(x^3 y^2 + x^2 y^3) & 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1 \\ (1/12)(8y^2 + 4y^2) & x > 2, 0 \leqslant y \leqslant 1 \\ (1/12)(x^3 + x^2) & y > 1, 0 \leqslant x \leqslant 2 \\ 1 & x > 2, y > 1 \end{cases}$$

b) [5 pts.] Compute the density $f_Y(y)$ of $Y$ alone.

$$f_Y(y) = \int_0^2 \frac{1}{2}(x^2 y + x y^2)\,\mathrm{d}x = \left[\frac{x^3 y}{6} + \frac{x^2 y^2}{4}\right]_0^2 = \frac{4}{3}y + y^2$$

c) [5 pts.] Compute the conditional probability density $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{(1/2)(x^2 y + x y^2)}{(4/3)y + y^2}$$

d) [5 pts.] Compute $P(X \leqslant 1 | Y = 2/3)$.

$$f_{X|Y}(x|2/3) = \frac{(1/2)(x^2(2/3) + x(4/9))}{(4/3)(2/3) + (4/9)} = \frac{x^2}{4} + \frac{x}{6}$$

$$P(X \leqslant 1 | Y = 2/3) = \int_0^1 (x^2/4 + x/6)\,\mathrm{d}x = \left.\frac{x^3 + x^2}{12}\right]_0^1 = \frac{1}{6}$$

## Problem 4

SKIP THIS PROBLEM □

Let $X$ be a continuous random variable with distribution $F(x) = \frac{x + |x| + 1}{2(|x| + 1)}$.

a) [6 pts] Compute $P(X \leqslant 0)$.

$$P(X \leqslant 0) = F(0) = \frac{1}{2}$$

b) [6 pts] Compute $P(1 \leqslant X \leqslant 3)$.

$$P(1 \leqslant X \leqslant 3) = F(3) - F(1) = \frac{7}{8} - \frac{3}{4} = \frac{1}{8}$$

c) [8 pts] Compute $P(X \geqslant 2 | X \geqslant 1)$.

$$P(X \geqslant 2 | X \geqslant 1) = \frac{1 - F(2)}{1 - F(1)} = \frac{1 - \frac{5}{6}}{1 - \frac{3}{4}} = \frac{1/6}{1/4} = \frac{2}{3}$$

## Problem 5

SKIP THIS PROBLEM ☐

A fair six-sided die is rolled 3600 times. Let $X$ be the number of 1s rolled; let $Y$ be the number of 6s rolled.

a) [5 pts] Compute $E[X]$ and $E[Y]$.

Since $X$,$Y$ are binomial with $n = 3600$, $p = 1/6$, $E[X] = E[Y] = 600$.

b) [5 pts] Compute $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$.

Since $X$,$Y$ are binomial with $n = 3600$, $p = 1/6$, $\mathrm{Var}[X] = \mathrm{Var}[Y] = 500$.

c) [4 pts] Compute $\mathrm{Var}[X + Y]$.

Since $X + Y$ is binomial with $n = 3600$, $p = 1/3$, $\mathrm{Var}[X + Y] = 800$.

d) [6 pts] Compute $\mathrm{Cov}[X, Y]$.

The fastest way to compute this is to use the following identity.

$$\mathrm{Var}[X+Y] = \mathrm{Cov}[X+Y, X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X, Y]$$

Using (b) and (c), $1000 = 800 + 2\mathrm{Cov}[X, Y]$, so $\mathrm{Cov}[X, Y] = -100$.

## Problem 6

SKIP THIS PROBLEM ☐

Alice and Betsy are playing a game involving a fair 12-sided die and twelve special cards. Each card corresponds to one of the sides of the dice. At the beginning of the game, Alice has all the cards. Each time the die is rolled, the matching card is passed from Alice to Betsy or from Betsy to Alice.

a) [7 pts] Compute the probability that, after six rolls of the die, Alice and Betsy each have six cards.

For this to happen, the die must indicate one of Alice's cards at each step. The probability is $(\frac{12}{12})(\frac{11}{12})(\frac{10}{12})(\frac{9}{12})(\frac{8}{12})(\frac{7}{12})$.

b) [5 pts] At a certain moment in the game, Alice has eight cards and Betsy has four. What is the probability that, before the previous die roll, Alice had nine cards?

If the most recent die roll was one of Betsy's four, then Alice previously had nine, otherwise not. The probability is 4/12=1/3.

c) [8 pts] At a certain other moment in the game, Alice has $X$ cards. After the next die roll, Alice has $Y$. After the next die roll, Alice has $Z$ cards. Compute the conditional probability that $Y = 9$, given that $X = Z = 8$.

Let $A_1$ be the event that Alice passes Betsy a card in the first of the two rolls considered here, and let $A_2$ be the event that Alice passes Betsy a card in the second of the two rolls. Let $B_1$, $B_2$ be likewise. We know that either $A_1B_2$ or $A_2B_1$ occurs (because Alice has the same number of cards after both passes as before).

We compute $P(A_1B_2) = (8/12)(5/12) = 40/144$ and $P(B_1A_2) = (4/12)(9/12) = 36/144$. Thus $P(B_1A_2|A_1B_2 \cup B_1A_2) = \frac{36/144}{40/144+36/144} = \frac{36}{76} = \frac{9}{19}$.

## Problem 7

SKIP THIS PROBLEM $\square$

Two cards are chosen from a standard, fairly shuffled deck of cards. Let $B$ be the event that both cards are aces. Let $H$ be the event that one of the cards is the ace of hearts.

a) [4 pts] Compute $P(B)$.

Of the $\frac{52 \cdot 51}{2}$ possible pairs, $\frac{4 \cdot 3}{2}$ are both aces. $P(B) = \frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{221}$.

b) [4 pts] Compute $P(H)$.

The ace of hearts is equally likely to be any of the 52 cards, so it is one of the two chosen with probability $\frac{1}{26}$.

c) [4 pts] Compute $P(BH)$.

Of the $\frac{52 \cdot 51}{2}$ possible pairs, 3 involve the ace of hearts and another ace. This gives $P(BH) = \frac{3}{(52 \cdot 51)/2} = \frac{1}{442}$.

d) [4 pts] Compute $P(B|H)$.

$$P(B|H) = \frac{P(BH)}{P(H)} = \frac{1/442}{1/26} = \frac{1}{17}$$

e) [4 pts] Compute $P(H|B)$.

$$P(H|B) = \frac{P(BH)}{P(B)} = \frac{1/442}{1/221} = \frac{1}{2}$$

## Problem 8

SKIP THIS PROBLEM □

Alice rolls a fair ten-sided die 1000 times, and Betsy flips a fair coin 200 times. Let $X$ be the number of 1s which Alice rolls, and let $Y$ be the number of heads which Betsy rolls. Think of $X$ as Alice's score, and $Y$ as Betsy's score. Alice wins if $X$ is more than $Y$.

a) [3 pts] Compute $E[X], E[Y]$.

Since $X$ is binomial with $n=1000$, $p=1/10$ and $Y$ is binomial with $n=200$, $p=1/2$, we have $E[X]=E[Y]=100$.

b) [3 pts] Compute $\mathrm{Var}[X], \mathrm{Var}[Y]$.

Since $X$ is binomial with $n=1000$, $p=1/10$ and $Y$ is binomial with $n=200$, $p=1/2$, we have $\mathrm{Var}[X]=90$ and $\mathrm{Var}[Y]=50$.

c) [7 pts] Estimate $P(80 \leqslant X \leqslant 120)$.

$X$ is approximately normal with $\mu=100, \sigma=\sqrt{90}$.

$$P(79.5 \leqslant X \leqslant 120.5) = P(-2.16 \leqslant \frac{X-\mu}{\sigma} \leqslant 2.16) = 2\Phi(2.16) - 1 = 0.9692$$

d) [7 pts] Estimate $P(\text{Alice wins})$.

$X-Y$ is approximately normal with $\mu=0, \sigma=\sqrt{140}$.

$$P(\text{Alice wins}) = P(X-Y \geqslant .5) = P(\frac{X-Y-\mu}{\sigma} \leqslant 0.042) = 1-\Phi(0.04) = 0.4840$$

# Problem 9

SKIP THIS PROBLEM ☐

The random variables $X$ and $Y$ are independent exponential random variables with rate $\lambda = 2$.

a) [7 pts] Compute $P(X > 3)$.

$$P(X>3) = \int_3^\infty 2e^{-2x}\mathrm{d}x = -e^{-2x}\Big]_3^\infty = e^{-6}$$

b) [7 pts] Find the probability density function of $X + Y$.

This is the convolution of $2e^{-2x}$ with itself.

$$f_{X+Y}(z) = \int_0^z f_X(x) f_Y(z-x)\mathrm{d}x = \int_0^z 4e^{-2z}\mathrm{d}x = 4ze^{-2z}$$

c) [6 pts] Compute $P(2 < X + Y < 4)$.

$$P(2 < X+Y < 4) = \int_2^4 4ze^{-2z}\mathrm{d}z = 5e^{-4} - 9e^{-8}$$

## Problem 10

SKIP THIS PROBLEM □

The $N = 2000000$ random variables $X_1, X_2, X_3, \ldots, X_N$ are independent and identically distributed with density function $f(x) = \begin{cases} x/2 & 0 \leqslant x \leqslant 2 \\ 0 & \text{else} \end{cases}$.

a) [5 pts] Compute $E[X_1]$.

$$E[X_1] = \int_0^2 \frac{x^2}{2} \mathrm{d}x = \frac{x^3}{6}\bigg]_0^2 = \frac{4}{3}$$

b) [5 pts] Compute $\mathrm{Var}[X_1]$.

$$E[X_1^2] = \int_0^2 \frac{x^3}{2} \mathrm{d}x = \frac{x^4}{8}\bigg]_0^2 = 2$$

$$\mathrm{Var}[X_1] = E[X_1^2] - E[X_1]^2 = 2 - (4/3)^2 = 2/9$$

c) [10 pts] Let $Y = \frac{1}{N}\sum_{i=1}^N X_i$ be the average of the $X_i$. Estimate $P(1.333 \leqslant Y \leqslant 1.334)$. (Assume that $N$ is large enough that the Central Limit Theorem gives an adequate estimate.)

First, $Y$ has mean $\mu = 4/3$ and standard devation $\sigma = \sqrt{2/9}/\sqrt{N} = \frac{1}{3000}$.

$$P(1.333 \leqslant Y \leqslant 1.334) = P\left(-1 \leqslant \frac{Y-\mu}{\sigma} \leqslant 2\right) = \Phi(2) + \Phi(1) - 1 = .9772 + .8413 - 1 = .8185$$