Final Exam (Sample Version #2)

Math 425-001 Sp10

by Prof. Michael "Cap" Khoury

Name:	

Directions:

Please print your name legibly in the box above.

You have 110 minutes to complete this exam. You may use any type of calculator without a QWERTY keyboard, but no computers/cell phones/iPads/etc. You may use three 3×5 note-cards (front and back) but no other reference materials of any kind. There are ten problems which are worth 20 points each. Partial credit is possible. *For full credit, clear and relevant work must be shown.*

You choose *nine* of the ten problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, then whichever problem is skipped by most students becomes the default skip. This is not what you want.

This exam is worth 180 points, representing 45% of the 400 total points possible in the course.

This exam consists of 6 pages, front and back (one side for this cover page, one side for a table of Φ, and one side for each of 10 problems). If you do not have all 6 pages or if any sides are blank, please notify me immediately!

Problem Number	Value	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
TOTAL	**160**	

Problem 1

SKIP THIS PROBLEM ☐

Gabriel is playing a solitaire game with a special deck of $(2n+2)$ cards. There are n matching pairs of cards with pictures of treasures, and one pair of cards labelled with skulls. All the cards are shuffled and laid out face down. Gabriel turns over cards, one at a time, trying to get pairs of treasures and avoid the skulls. Whenever he gets a matching pair of treasures, he gets a point. He turns over cards until he completes the pair of skulls; then he has to stop. (Note: it doesn't matter whether matching pairs are turned over on consecutive flips or not.)

a) [10 pts] Compute the probability that Gabriel will find all n pairs of treasures.

Gabriel will find all the treasures precisely if the very last card he turns up is a skull, which happens with probability $\frac{2}{2n+2} = \frac{1}{n+1}$.

b) [10 pts] Compute Gabriel's expected score.

Let X_i be the indicator of E_i, the event that Gabriel finds both instances of treasure i (for $i = 1, 2, ..., n$). Then $E[\text{score}] = \sum E[X_i] = \sum P(E_i)$. Now, Gabriel finds treasure i if, among the two copies of treasure i and the two skulls, he finds a skull last: $P(E_i) = \frac{1}{2}$.

$$E[\text{score}] = \sum_{i=1}^{n} P(E_i) = \frac{n}{2}$$

Problem 2

SKIP THIS PROBLEM □

The $N=30000$ random variables $X_1, X_2, X_3, \ldots, X_N$ are independent and identically distributed with density function $f(x)=\begin{cases} 24/x^4 & x>2 \\ 0 & \text{else} \end{cases}$.

a) [5 pts] Compute $E[X_1]$.

$$E[X_1]=\int_2^\infty \frac{24}{x^3}\mathrm{d}x == -12x^{-2}\Big]_2^\infty = 3$$

b) [5 pts] Compute $\mathrm{Var}[X_1]$.

$$E[X_1^2]=\int_2^\infty \frac{24}{x^2}\mathrm{d}x = -24x^{-1}\Big]_2^\infty = 12$$

$$\mathrm{Var}[X_1]=E[X_1^2]-E[X_1]^2=12-3^2=3$$

c) [10 pts] Let $Y=\frac{1}{N}\sum_{i=1}^N X_i$ be the average of the X_i. Estimate $P(3.01 \leqslant Y \leqslant 3.015)$. (Assume that N is large enough that the Central Limit Theorem gives an adequate estimate.)

First, Y has mean $\mu=3$ and standard devation $\sigma=\sqrt{3}/\sqrt{N}=0.01$.

$$P(3.01 \leqslant Y \leqslant 3.015)=P\left(1 \leqslant \frac{Y-\mu}{\sigma} \leqslant 1.5\right)=\Phi(1.5)-\Phi(1)=.9332-.8413=.0919$$

Problem 3

SKIP THIS PROBLEM □

Let X be a continuous random variable with density $f(x) = k(x^2 - x)$ for $1 < x < 2$ (and 0 elsewhere).

a) [6 pts] Compute k.

We must have $1 = \int_1^2 f(x)\,\mathrm{d}x = \int_1^2 k(x^2 - x)\,\mathrm{d}x = k\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = \frac{5}{6}k$, so $k = \frac{6}{5}$.

b) [7 pts] Compute $P(X < \frac{3}{2})$.

$$P(X < 3/2) = \frac{6}{5}\int_1^{3/2} (x^2 - x)\,\mathrm{d}x = \left[\frac{2x^3}{5} - \frac{3x^2}{5}\right]_1^{3/2} = \frac{1}{5}$$

c) [7 pts] Compute $P\left(X > \frac{5}{4}\middle| X < \frac{3}{2}\right)$.

$$\begin{aligned} P\left(X > \frac{5}{4}\middle| X < \frac{3}{2}\right) &= \frac{P(5/4 < X < 3/2)}{P(X < 3/2)} \\ &= 6\int_{5/4}^{3/2} (x^2 - x)\,\mathrm{d}x \\ &= \frac{25}{32} \end{aligned}$$

Problem 4

SKIP THIS PROBLEM ☐

The continuous random variables X, Y, Z are independent. Their density functions are given by $f_X(x) = \left(\frac{1}{e-1}\right)e^{1-x}$, $f_Y(y) = 2y$, $f_Z(z) = 2z$. All of these are defined only for $0 \leqslant x, y, z \leqslant 1$.

a) [2 pts] Write down the joint density function $f_{X,Y,Z}(x, y, z)$, where $0 \leqslant x, y, z \leqslant 1$.

$$f_{X,Y,Z}(x,y,z) = \frac{4}{e-1} e^{1-x} yz$$

b) [5 pts] Write down the joint distribution function $F_{X,Y,Z}(x, y, z)$, where $0 \leqslant x, y, z \leqslant 1$.

For x, y, z in that range, $F_X(x) = \frac{e}{e-1}(1 - e^{-x})$, $F_Y(y) = y^2$, $F_Z(z) = z^2$.

$$F_{X,Y,Z}(x,y,z) = \frac{e}{e-1}(1 - e^{-x}) y^2 z^2$$

c) [8 pts] Compute $P(X > Y)$.

$$\begin{aligned} P(X > Y) &= \int_0^\infty \int_0^x \frac{2}{e-1} e^{1-x} y \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^\infty \frac{1}{e-1} x^2 e^{1-x} \, \mathrm{d}x \\ &= -\left. \frac{e^{1-x}(x^2 + 2x + 2)}{e-1} \right]_0^1 \\ &= \frac{2e - 5}{e-1} \end{aligned}$$

d) [5 pts] Compute $P(Y > Z)$.

Y and Z are continuous, independent, and identically distributed, so without any computation $P(Y > Z) = P(Y < Z) = 1/2$.

Problem 5

SKIP THIS PROBLEM □

As part of a certain obscure board game, I roll a special fair 12-sided die. Of the die's twelve sides, four are marked with stars, three are marked with hearts, two are marked with circles, two are marked with lightning bolts, and one is marked with a question mark. Let S, H, C, L, Q be the number of stars, hearts, circles, lightning, and question marks rolled in 10 independent rolls of the die.

a) [6 pts] Compute $P(S=2, H=1, C=4, L=1, Q=2)$.

$$\left(\frac{10!}{2!1!4!1!2!}\right)\left(\frac{4}{12}\right)^2\left(\frac{3}{12}\right)^1\left(\frac{2}{12}\right)^4\left(\frac{2}{12}\right)^1\left(\frac{1}{12}\right)^2$$

b) [6 pts] Compute $P(C=0)$.

The probability that any individual roll is not a circle is 5/6, so $P(C=0)=(5/6)^{10}$.

c) [8 pts] Compute $P(S+Q>H+C+L)$.

First, $P(S+Q>H+C+L)=P(S+Q\geqslant 6)$. Also, $S+Q$ is itself a binomial variable with $n=10$ and $p=5/12$.

$$210\left(\frac{5}{12}\right)^6\left(\frac{7}{12}\right)^4+120\left(\frac{5}{12}\right)^7\left(\frac{7}{12}\right)^3+45\left(\frac{5}{12}\right)^8\left(\frac{7}{12}\right)^2+10\left(\frac{5}{12}\right)^9\left(\frac{7}{12}\right)+\left(\frac{5}{12}\right)^{10}$$

Problem 6

SKIP THIS PROBLEM ☐

A professor occasionally employs any of three students to do typing and other such work. Assume that the number of errors made by any of the students in typing is a Poisson random variable. Alice averages 1 error per thousand words, Betsy averages 2 errors per thousand words, and Claire averages 5 errors per thousand words.

The professor needs a 10,000-word document typed up, and she will give the task to whichever student shows up first, which will be Alice with probability $2/5$, Betsy with probability $2/5$, or Claire with probability $1/5$.

Let X be the number of errors made in typing the document.

Let A, B, C be the events that Alice, Betsy, and Claire type the document (respectively).

a) [5 pts] Compute $E[X]$.

First, we have $E[X|\text{Alice}] = 10$, $E[X|\text{Betsy}] = 20$, $E[X|\text{Claire}] = 50$. Then we compute $E[X] = 10(2/5) + 20(2/5) + 50(1/5) = 22$.

b) [5 pts] Find $P(X = 15)$.

$$\begin{aligned} P(X=15) &= P(X=15|A)P(A) + P(X=15|B)P(B) + P(X=15|C)P(C) \\ &= e^{-10}\frac{10^{15}}{15!}\left(\frac{2}{5}\right) + e^{-20}\frac{20^{15}}{15!}\left(\frac{2}{5}\right) + e^{-50}\frac{50^{15}}{15!}\left(\frac{1}{5}\right) \end{aligned}$$

c) [10 pts] Given that there are exactly 15 errors, give an expression for the conditional probability that Alice typed the document.

$$P(A|X=15) = \frac{P(A, X=15)}{P(X=15)} = \frac{e^{-10}\frac{10^{15}}{15!}\left(\frac{2}{5}\right)}{e^{-10}\frac{10^{15}}{15!}\left(\frac{2}{5}\right) + e^{-20}\frac{20^{15}}{15!}\left(\frac{2}{5}\right) + e^{-50}\frac{50^{15}}{15!}\left(\frac{1}{5}\right)}$$

This is adequate as it stands, but we can clean it up a little if we so desire.

$$P(A|X=15) = \frac{2e^{40}}{2e^{40} + 2e^{30}2^{15} + 5^{15}}$$

Problem 7

SKIP THIS PROBLEM ☐

Let X_1, X_2 be uniform continuous random variables on the interval $[0,\ 10]$. Let Y_1, Y_2 be the smaller and larger of $X_{1,}$, X_2, respectively.

a) [5 pts] Compute $E[Y_1]$.

$Y_1 = X_{(1)}$ has density function $f(y_1) = 2(\frac{1}{10})(\frac{10-y_1}{10}) = \frac{1}{50}(10-y_1)$

$$E[Y_1] = \int_0^{10} y_1 \frac{1}{50}(10-y_1)\mathrm{d}y_1 = \frac{10}{3}$$

b) [5 pts] Compute $E[Y_2]$.

$Y_2 = X_{(2)}$ has density function $f(y_2) = 2(\frac{1}{10})(\frac{y_2}{10}) = \frac{1}{50}(y_2)$

$$E[Y_2] = \int_0^{10} y_2 \frac{1}{50} y_2 \mathrm{d}y_2 = \frac{20}{3}$$

c) [5 pts] Compute $E[Y_1 Y_2]$.

$Y_1, Y_2 = X_{(1)}, X_{(2)}$ have joint density function $f(y_1, y_2) = \frac{1}{50}$, supported on the triangular region $0 \leqslant Y_1 \leqslant Y_2 \leqslant 10$.

$$E[Y_1 Y_2] = \int_0^{10} \int_0^{y_2} y_1 y_2 \frac{1}{50} \mathrm{d}y_1 \mathrm{d}y_2 = \int_0^{10} \frac{1}{100} y_2^3 \, \mathrm{d}y_2 = 25$$

d) [5 pts] Compute $\mathrm{Cov}[Y_1, Y_2]$.

$$\mathrm{Cov}[Y_1, Y_2] = E[Y_1 Y_2] - E[Y_1]E[Y_2] = 25 - (20/3)(10/3) = 25/9$$

Problem 8

SKIP THIS PROBLEM ☐

The continuous random variables X, Y have a joint uniform distribution on the triangular region $0 \leqslant X \leqslant Y \leqslant 4$.

a) [5 pts] Compute $P(Y \geqslant 2)$.

This can be done by geometry. The support of X, Y has area 8, and the trapezoidal region with $Y \geqslant 2$ has area 6. The probability is $3/4$.

b) [5 pts] Compute the density $f_X(x)$ of X alone.

First, the joint density is $1/8$ where it is supported.

$$f_X(x) = \int_x^4 \frac{1}{8} \mathrm{d}y = \frac{4-x}{8}$$

c) [5 pts] Compute the conditional probability density $f_{Y|X}(y|x)$.

First, the joint density is $1/8$ where it is supported.

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/8}{(4-x)/8} = \frac{1}{4-x}.$$

d) [5 pts] Compute $E[Y|X=x]$, where $0 < x < 4$.

$$E[Y|X=x] = \int_x^4 y f_{Y|X}(y|x) \mathrm{d}y = \int_x^4 \frac{y}{4-x} \mathrm{d}y = \frac{4-x}{2}$$

(We don't really need an integral for this. Given $X = x$, Y is conditionally uniformly distributed on $[x, 4]$, and the expectation is the midpoint of the interval.)

Problem 9

SKIP THIS PROBLEM □

In a certain game, the player rolls three dice at a time repeatedly, stopping the first time all three dice turn up different numbers. Let X be the number of tries necessary to roll three different numbers, and let Y be the largest of the three numbers on the final die roll.

a) [6 pts] Compute $P(X = 1)$.

Of the $6^3 = 216$ equally likely possible die rolls, $6 \cdot 5 \cdot 4 = 120$ have three different numbers. The probability of the first die roll being having three different numbers is $\frac{120}{216} = \frac{5}{9}$.

b) [6 pts] Find $E[X]$.

Since X is geometric with $p = 5/9$, $E[X] = 9/5$.

c) [8 pts] Compute $P(Y = 6)$.

Of the $\binom{6}{3} = 20$ equally likely possible sets of three different die rolls, $\binom{5}{2} = 10$ include a 6, so $P(Y = 6) = 1/2$.

Problem 10

SKIP THIS PROBLEM □

Let X be a uniform variable on the interval $[0, 1]$, and define $Y = X^3$.

a) [10 pts] Compute the density function of Y.

We compute, for $0 < y < 1$, $F_Y(y) = P(Y \leqslant y) = P(X \leqslant y^{1/3}) = y^{1/3}$. Then we differentiate to obtain $f_Y(y) = \frac{1}{3}y^{-2/3}$. (For $y \leqslant 0$ or $y \geqslant 1$, $f_Y(y) = 0$ of course.)

b) [5 pts] Compute $E[Y]$.

$$E[Y] = \int_0^1 y f_Y(y) \mathrm{d}y = \int_0^1 \frac{1}{3} y^{1/3} \mathrm{d}y = \frac{1}{4} y^{4/3} \Big]_0^1 = 1/4$$

c) [5 pts] Compute $E[Y^2]$.

$$E[Y^2] = \int_0^1 y^2 f_Y(y) \mathrm{d}y = \int_0^1 \frac{1}{3} y^{4/3} \mathrm{d}y = \frac{1}{7} y^{7/3} \Big]_0^1 = 1/7$$