Final Exam

Math 425-001 Sp10

by Prof. Michael "Cap" Khoury

Name:	

Directions:

Please print your name legibly in the box above.

You have 110 minutes to complete this exam. You may use any type of calculator without a QWERTY keyboard, but no computers/cell phones/iPads/etc. You may use three 3×5 notecards (front and back) but no other reference materials of any kind. There are ten problems which are worth 20 points each. Partial credit is possible. *For full credit, clear and relevant work must be shown.*

You choose *nine* of the ten problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, then whichever problem is skipped by most students becomes the default skip. This is not what you want.

This exam is worth 180 points, representing 45% of the 400 total points possible in the course.

This exam consists of 6 pages, front and back (one side for this cover page, one side for a table of Φ, and one side for each of 10 problems). If you do not have all 6 pages or if any sides are blank, please notify me immediately!

Problem Number	Value	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
TOTAL	**180**	

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Figure 1.

Problem 1

SKIP THIS PROBLEM ☐

Gabriel has a supply of n balls, and his sister Datura has a supply of n boxes. They play the following game. Datura places a box into the middle of the room; Gabriel puts a ball into the box. Datura places a second box into the middle of the room; Gabriel puts a ball into one of the two boxes. Datura places a third box into the middle of the room; Gabriel puts a ball into one of the three boxes. ...and so on, until... Datura places her nth box into the middle of the room, and then Gabriel puts his nth ball into one of the n boxes.

That is, when Gabriel places his kth ball, he has k boxes to choose from. There is no limit to the number of balls that may end up in any one box. Assume that whenever he has a choice of boxes, Gabriel makes the choice uniformly randomly, independently from all other choices.

a) [10 pts] What is the probability that, at the end of the game, none of the boxes are empty?

There will be no empty boxes if and only if Gabriel always puts his ball in the newest box. The probability that this will happen is $\frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \dots \cdot \frac{1}{n} = \frac{1}{n!}$,

b) [10 pts] Compute the expected number of empty boxes at the end of the game.

Let X_i be the indicator of E_i, the event that the ith box is empty at the end of the game. Then E_i happens if and only if Gabriel avoids box i at turns i, $i+1$, $i+2$, ..., n. Thus $E[X_i] = P(E_i) = \frac{i-1}{i} \cdot \frac{i}{i+1} \cdot \dots \cdot \frac{n-1}{n} = \frac{i-1}{n}$, and $E[\#\text{empty boxes}] = \sum_{i=1}^{n} \frac{i-1}{n} = \frac{n(n-1)}{2}$.

Problem 2

SKIP THIS PROBLEM ☐

The random variable X has a beta distribution; its density is given by

$$f(x) = \begin{cases} kx^2(1-x) & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases},$$

where k is some positive constant.

a) [6 pts] What is k?

$$\int_0^1 kx^2(1-x)\mathrm{d}x = k\int_0^1 (x^2 - x^3)\mathrm{d}x = k/12$$

This must equal 1, so $k = 12$.

b) [7 pts] Compute $P\left(X \leqslant \frac{1}{2}\right)$.

$$\int_0^{1/2} 12x^2(1-x)\mathrm{d}x = \left[4x^3 - 3x^4\right]_0^{1/2} = \frac{4}{8} - \frac{3}{16} = \frac{5}{16}$$

c) [7 pts] Compute $P\left(X \leqslant \frac{1}{3} \middle| X \leqslant \frac{1}{2}\right)$.

$$\int_0^{1/2} 12x^2(1-x)\mathrm{d}x = \left[4x^3 - 3x^4\right]_0^{1/2} = \frac{4}{8} - \frac{3}{16} = \frac{5}{16}$$

$$\int_0^{1/3} 12x^2(1-x)\mathrm{d}x = \left[4x^3 - 3x^4\right]_0^{1/3} = \frac{4}{27} - \frac{3}{81} = \frac{1}{9}$$

$$P\left(X \leqslant \frac{1}{3} \middle| X \leqslant \frac{1}{2}\right) = \frac{P(X \leqslant 1/3)}{P(X \leqslant 1/2)} = \frac{1/9}{5/16} = \frac{16}{45}$$

Problem 3

SKIP THIS PROBLEM □

The integer-valued random variables X, Y have a joint discrete distribution, with the following probability mass function.

$$P(X=n, Y=m) = \begin{cases} 1/n^m & n, m \geqslant 2 \\ 0 & \text{else} \end{cases}$$

a) [6 pts] Compute $P(X=n)$ for $n \geqslant 2$.

The series involved here is geometric. $P(X=n) = \sum_{m=2}^{\infty} (1/n)^m = \frac{1/n^2}{1-1/n} = \frac{1}{n(n-1)}$

b) [7 pts] Compute $P(X=3|X \geqslant 3)$.

$P(X=3) = \frac{1}{6}, P(X=2) = \frac{1}{2}$, and $P(X \geqslant 3) = 1 - P(X=2) = \frac{1}{2}$.

So $P(X=3|X \geqslant 3) = \frac{1/6}{1/2} = \frac{1}{3}$.

c) [7 pts] Compute $P(Y=3|X=3)$.

$$P(Y=3|X=3) = \frac{P(X=3, Y=3)}{P(X=3)} = \frac{1/27}{1/6} = \frac{2}{9}$$

Problem 4

SKIP THIS PROBLEM ☐

Ashley and Joe are meeting for coffee. Each arrives at a time between noon and 1pm, uniformly on that interval and independently from one another. Let X be Ashley's arrival time (measured in minutes after noon), and let Y be the corresponding thing for Joe. Let Z be the amount of time (in minutes) that whoever arrives first has to wait for the other.

a) [6 pts] Compute $P(Z < 20)$.

The space of outcomes is naturally viewed as a square of side length 60. The event that Ashley waits more than k minutes $(0 < k < 60)$ is naturally viewed as a right triangle with legs of length $(60 - k)$. So the combined event that Joe or Ashley waits at least k minutes has area $(60 - k)^2$. The probability $P(Z < 20)$, then, is $\frac{60^2 - 40^2}{60^2} = \frac{5}{9}$.

b) [7 pts] Compute $E[Z]$.

$$\begin{aligned} E[Z] &= \frac{1}{3600} \int_0^{60} \int_0^{60} |x - y| \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{3600} \left(\int_0^{60} \int_y^{60} (x - y) \mathrm{d}x \, \mathrm{d}y + \int_0^{60} \int_0^{y} (y - x) \mathrm{d}x \, \mathrm{d}y \right) \end{aligned}$$

Actually, since both of these integrals are easily seen to give the same answer, it's enough to compute either. In any case, $E[Z] = 20$.

c) [7 pts] Compute the conditional expectation of Y, given that Ashley is the one who arrives first.

$$E[Y] = \frac{1}{1800} \int_0^{60} \int_0^{y} y \, \mathrm{d}x \, \mathrm{d}y = 40$$

Problem 5

SKIP THIS PROBLEM ☐

The random variables X, Y are jointly continuously distributed with joint density given by

$f(x,y) = \begin{cases} x^{-3}e^{-y} + x^{-2}e^{-2y} & x \geqslant 1;\, y \geqslant 0 \\ 0 & \text{else} \end{cases}.$

a) [5 pts.] Compute the joint distribution function $F(x, y)$.

For $x \geqslant 1, y \geqslant 0$, $F(x,y) = \int_1^x \int_0^y (x'^{-3}e^{-y'} + x'^{-2}e^{-2y'})\mathrm{d}x'\,\mathrm{d}y' = \frac{1}{2}(x^{-2}e^{-y} + x^{-1}e^{-2y})$.

b) [5 pts.] Compute the density $f_Y(y)$ of Y alone.

$$f_Y(y) = \int_1^\infty (x^{-3}e^{-y} + x^{-2}e^{-2y})\mathrm{d}x = \frac{1}{2}e^{-y} + e^{-2y}$$

c) [5 pts.] Compute the conditional probability density $f_{X|Y}(x|y)$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x^{-3}e^{-y} + x^{-2}e^{-2y}}{\frac{1}{2}e^{-y} + e^{-2y}}$$

d) [5 pts.] Compute $P(X \leqslant 2 | Y = 1)$.

$$\int_1^2 \frac{x^{-3}e^{-1} + x^{-2}e^{-2}}{\frac{1}{2}e^{-1} + e^{-2}}\mathrm{d}x = \frac{\frac{7}{8}e^{-1} + \frac{3}{4}e^{-2}}{\frac{1}{2}e^{-1} + e^{-2}}$$

Problem 6

SKIP THIS PROBLEM ☐

Let X_1, X_2 be independent exponentially distributed random variables, each with rate $\lambda = 8$. Let $Y = X_1 + X_2, Z = X_1 - X_2$.

a) [4 pts] Compute $P(Z > 0)$.

$P(X_1 > X_2) = P(X_2 > X_1)$ by symmetry, so $P(Z > 0) = 1/2$.

b) [4 pts] Compute $E[Y]$.

$E[Y] = E[X_1] + E[X_2] = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

c) [4 pts] Compute $E[Z]$.

$E[Z] = E[X_1] - E[X_2] = 0$.

d) [4 pts] Compute $E[YZ]$.

$YZ = (X_1 + X_2)(X_1 - X_2) = X_1^2 - X_2^2$, so $E[YZ] = E[X_1^2] - E[X_2^2] = 0$.

e) [4 pts] Compute $\mathrm{Cov}[Y, Z]$.

$\mathrm{Cov}[Y, Z] = E[YZ] - E[Y]E[Z] = 0$

Problem 7

SKIP THIS PROBLEM ☐

Alice and Betsy are playing a game involving some specialty dice. These dice are fair six-diced dice, but instead of numbers the sides have colors. Each die has three purple sides, two red sides, and a gold side. Simultaneously and independently, Alice rolls one of these dice and Betsy rolls twelve. Let P be the number of purples showing on Betsy's dice, let R be the number of reds showing on Betsy's dice, and let G be the number of golds showing on Betsy's dice. Also let X be the number of Betsy's dice showing the same color that Alice's die shows.

a) [5 pts] Write down an expression for $P(P=5, R=4, G=3)$.

$$\binom{12}{5,4,3}\left(\frac{1}{2}\right)^5\left(\frac{1}{3}\right)^4\left(\frac{1}{6}\right)^3$$

b) [5 pts] Compute $P(R=4)$.

Note that R is itself binomial with $p=1/3$.

$$\binom{12}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^8$$

c) [10 pts] Compute the conditional probability that Alice rolled red, given that $X=4$.

Let A be the color of Alice's die. Then we have $P(A=\text{purple})=1/2$; $P(A=\text{red})=1/3$, and $P(A=\text{gold})=1/6$.

$$P(X=4, A=\text{purple}) = (1/2)P(P=4) = \binom{12}{4}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^8$$

$$P(X=4, A=\text{red}) = (1/3)P(R=4) = \binom{12}{4}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^8$$

$$P(X=4, A=\text{gold}) = (1/2)P(G=4) = \binom{12}{4}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^8$$

$$P(A=\text{red}|X=4) = \frac{\binom{12}{4}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^8}{\binom{12}{4}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^8+\binom{12}{4}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^8+\binom{12}{4}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^8}$$

Problem 8

SKIP THIS PROBLEM ☐

Assume that the number of accidents on a particular stretch of highway on a given day is a Poisson random variable with $\lambda = 3$. If there are no accidents on a given day, we call that a *lucky* day. Days that are not lucky (days with at least one accident) are called *unlucky*. Days with at least three accidents are called *extremely unlucky*.

Let X be the number of accidents on a given day.

a) [6 pts] Compute the probability that a given day is lucky.

$P(X=0)=e^{-3}\frac{3^0}{0!}=e^{-3}$.

b) [6 pts] Compute the probability that a given day is extremely unlucky.

$P(X\geqslant 3)=1-P(X=0)-P(X=1)-P(X=2)=1-e^{-3}(\frac{3^0}{0!}+\frac{3^1}{1!}+\frac{3^2}{2!})=1-\frac{17}{2}e^{-3}$.

c) [8 pts] Compute the conditional probability that a given day is extremely unlucky, given that it is unlucky.

$P(X\geqslant 3|X>0)=\frac{P(X\geqslant 3)}{1-P(X=0)}=\frac{1-\frac{17}{2}e^{-3}}{1-e^{-3}}=\frac{2e^3-17}{2e^3-2}$.

Problem 9

SKIP THIS PROBLEM ☐

The $N = 30000$ random variables $X_1, X_2, X_3, \ldots, X_N$ are independent and uniformly distributed on the interval $[1, 3]$.

a) [5 pts] Compute $E[X_1]$.

$E[X_1] = 2$

b) [5 pts] Compute $\mathrm{Var}[X_1]$.

$\mathrm{Var}[X_1] = (3-1)^2/12 = 1/3$

c) [10 pts] Let $Y = \frac{1}{N}\sum_{i=1}^{N} X_i$ be the average of the X_i. Estimate $P(2.009 \leqslant Y \leqslant 2.01)$. (Assume that N is large enough that the Central Limit Theorem gives an adequate estimate.)

The random variable Y is approximately normal with $\mu = 2$, $\sigma = \frac{1/\sqrt{3}}{\sqrt{N}} = \frac{1}{300}$.

$$P(2.009 \leqslant Y \leqslant 2.01) = P\left(2.7 \leqslant \frac{Y-\mu}{\sigma} \leqslant 3.0\right) \approx \Phi(3) - \Phi(2.7) \approx 0.0022$$

Problem 10

SKIP THIS PROBLEM □

In a certain society, every family that has children at all has children until there is at least one son and and at least one daughter, then stops. (Assume that there are no multiple births, and that any child born is equally likely to be a boy or a girl, independently of all other children.)

a) [3 pts] What is the probability that the eldest child in a family with children is a girl?

Trick question. The probability is $1/2$, the same as for a boy.

b) [3 pts] What is the probability that the youngest child in a family with children is a girl?

Trick question. The probability is $1/2$, the same as for a boy.

c) [5 pts] Compute the probability that a family with children has exactly three children?

If a family has children, then the number of children they have is $1+X$, where X is geometric with $p=1/2$.

So $P(3\,\text{children}|\text{children}) = P(X=2) = \frac{1}{4}$.

d) [9 pts] It turns out that the expected number of children in a family in this society is 2. What is the probability that a given family in this society has no children?

By the solution method to part (c), the expected number of children in a family with children is 3. Let p be the probability that a given family in this society has no children. Then $2 = E[\#\text{children}] = (p)(0) + (1-p)3$, leading to $p = 1/3$.