Midterm (Sample Version 1, with Solutions)

Math 425-201 Su10

by Prof. Michael "Cap" Khoury

Name:	

Directions:

Please print your name legibly in the box above.

You have 110 minutes to complete this exam. You may use any type of calculator without a QWERTY keyboard, but no computers/cell phones/iPads/etc. You may use two 3×5 notecards (front and back) but no other reference materials of any kind. There are eleven problems which are worth 16 points each. Partial credit is possible. *For full credit, clear and relevant work must be shown.*

You choose *ten* of the eleven problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, then whichever problem is skipped by most students becomes the default skip. This is not what you want.

This exam is worth 160 points, representing 40% of the 400 total points possible in the course.

This exam consists of 6 pages, front and back (one side for this cover page and one side for each of 11 problems). If you do not have all 6 pages or if any sides are blank, please notify me immediately!

Problem Number	Value	Score
1	16	
2	16	
3	16	
4	16	
5	16	
6	16	
7	16	
8	16	
9	16	
10	16	
11	16	
TOTAL	**160**	

Problem 1

SKIP THIS PROBLEM □

Let X be a discrete random variable which can take the values 1, 2, 3, ..., with the following probabilities.

$$P(X = n) = \frac{1}{n(n+1)} \qquad n = 1, 2, 3, \ldots$$

a) Compute $P(X \leq 4)$.

$$\begin{aligned} P(X \leqslant 4) &= P(X=1) + P(X=2) + P(X=3) + P(X=4) \\ &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \\ &= \frac{4}{5} \end{aligned}$$

b) Compute $P(X \geq 2)$.

$$\begin{aligned} P(X \geqslant 2) &= 1 - P(X=1) \\ &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

c) Compute $P(X = 3 | X \neq 1)$.

$$\begin{aligned} P(X=3) &= \frac{P(X=3 \text{ and } X \neq 1)}{P(X \neq 1)} \\ &= \frac{P(X=3)}{P(X \neq 1)} \\ &= \frac{1/12}{1/2} \\ &= 1/6 \end{aligned}$$

Problem 2

SKIP THIS PROBLEM ☐

I have a special deck of 90 cards. The cards are numbered from 1–15, and there are six different suits (hearts, spades, diamonds, clubs, stars, and spiders).

a) How many different five-card hands are there?

$\binom{90}{5} = \frac{90 \cdot 89 \cdot 88 \cdot 87 \cdot 86}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 43949268$

b) How many different five-card hands are there in which all five cards are of the same suit?

There are 6 ways to choose the suit, then $\binom{15}{5}$ ways to choose the cards.

$6\binom{15}{5} = 6\frac{15!}{5!10!} = 18018$

c) How many different five-card hands are there in which all five cards are different suits and different numbers?

For the first card, there are 6 possible suits and 15 possible ranks; for the second card, there are 5 possible suits and 14 possible ranks, etc. However, this overcounts because it counts each hand $5! = 120$ times.

$\frac{6 \cdot 15 \cdot 5 \cdot 14 \cdot 4 \cdot 13 \cdot 3 \cdot 12 \cdot 2 \cdot 11}{5!} = 2162160$

Problem 3

SKIP THIS PROBLEM ☐

There is a circular table with eight equally-spaced seats. Six people, including Smith and Jones, sit down in random seats, with each possible seating arrangement equally likely. (Note that there will be two empty seats.) Let E be the event that Smith is sitting in a seat next to Jones, let F be the event that someone is "isolated" (immediately between the two empty seats).

a) Evaluate $P(E)$.

Of the seven seats not taken by Smith (which might be taken by Jones), two are next to Smith, so $P(E) = \frac{2}{7}$.

b) Evaluate $P(F)$.

The probability that the empty seats are on either side of Smith is $\frac{1}{\binom{7}{2}} = \frac{1}{21}$. Because there are six people who might be the isolated one, $P(F) = \frac{6}{21} = \frac{2}{7}$.

Problem 4

SKIP THIS PROBLEM ☐

In the game of chess, each player has 8 pawns, 2 bishops, 2 knights, 2 rooks, 1 queen, and 1 king. In standard chess, this 16 pieces are placed in a standard configuration on the first two rows of a chessboard. In a certain variation, each player can place the 16 pieces on those 16 squares in any arrangement she wants.

a) How many arrangements are possible?

The number of arrangements is the given by the multinomial coefficient.

$\binom{16}{8,2,2,2,1,1} = \frac{16!}{8!2!2!2!1!1!} = 64864800$

b) How many arrangements are possible if the king must be somewhere in the back row?

The king is just as likely to be in the front row as in the back row, so half of the arrangements are still legitimate. There are now 32432400 arrangements.

Problem 5

SKIP THIS PROBLEM ☐

Suppose that the number of "wrong number" phone calls which I receive in a day is a Poisson variable with mean 1, and suppose further that the number of wrong numbers on different days are independent.

Let X be the number of days until the first day without any wrong numbers. (If there are no wrong numbers today, X takes the value 1; if there is at least one wrong number today but no wrong numbers tomorrow, $X=2$, etc.)

a) Compute $P(X=1)$.

Let Y be a Poisson random variable with mean 1. The probability that there are no wrong numbers today is $P(Y=0)=e^{-1}$.

b) Compute $E[X]$.

Since the probability of no wrong numbers on any given day is e^{-1}, X is a geometric random variable with $p=e^{-1}$. $E[X]=1/p=e$.

Problem 6

SKIP THIS PROBLEM □

I have three six-sided dice, indistinguishable to the naked eye.

- an ordinary fair die
- my "lucky" die, which comes up 4, 5, or 6 each with probability 1/3 (and never comes up 1, 2, nor 3).
- my "unlucky" die, which comes up 1, 2, or 3, each with probability 1/3 (and never comes up 4, 5, nor 6).

I pick up one of these dice at random (each die equally likely to be chosen) and roll it repeatedly. Let A be the event that I choose the fair die, let E_2 be the event that my first two rolls give the same number, and let E_3 be the event that my first three rolls all give the same number.

a) Compute $P(E_2)$.

$$\begin{aligned} P(E_2) &= P(E_2|A)P(A) + P(E_2|A^c)P(A^c) \\ &= (1/6)(1/3) + (1/3)(2/3) \\ &= 5/18 \end{aligned}$$

b) Compute $P(A|E_2)$.

$$\begin{aligned} P(A|E_2) &= \frac{P(E_2|A)P(A)}{P(E_2)} \\ &= \frac{(1/6)(1/3)}{5/18} \\ &= 1/5 \end{aligned}$$

c) Compute $P(E_3|E_2)$.

$$\begin{aligned} P(E_3|E_2) &= P(E_3|AE_2)P(A|E_2) + P(E_3|A^cE_2)P(A^c|E_2) \\ &= (1/6)(1/5) + (1/3)(4/5) \\ &= 3/10 \end{aligned}$$

Problem 7

SKIP THIS PROBLEM ☐

If A, B are dependent events with $P(B) > 0$, then we say that B *gives positive information about* A if $P(A|B) > P(A)$ and that B *gives negative information about* A if $P(A|B) < P(A)$.

a) Suppose I have shuffled an ordinary 52-card deck. Let E be the event that the top card is an ace, and let F be the even that the top card is a heart. Are E, F independent? If not, does F give positive or negative information about E?

   $P(E) = 1/13$, $P(F) = 1/4$, $P(EF) = 1/52$. The events E, F are independent.

b) Now suppose I add two jokers to make a 54-card deck. (Jokers have no rank, and belong to no suit.) Again let E be the event that the top card is an ace, and let F be the even that the top card is a heart. Are E, F independent now? If not, does F give positive or negative information about E?

   $P(E) = 2/27$, $P(F) = 13/54$, $P(EF) = 1/54$. The events E, F are dependent.

   Since $P(E|F) = \frac{1/54}{13/54} = \frac{1}{13} > \frac{2}{27} = P(E)$, F gives positive information about E. (This makes sense, since knowing that a card is heart means that it is not the joker, hence more likely to be a heart.)

Problem 8

SKIP THIS PROBLEM ☐

A club of 15 people forms two 5-person committees. Each committee is formed randomly and independently, with every choice of 5 people equally likely to be chosen. Note that it is perfectly possible that the two commitees overlap. Let X be the number of people who are on both commitees (so that $0 \leq X \leq 5$).

a) Compute $P(X=0)$.

There are $\binom{15}{5}$ possible second committees, of which $\binom{10}{5}$ are disjoint from the first committee.

$P(X=0) = \frac{\binom{10}{5}}{\binom{15}{5}} = \frac{12}{143}$.

b) Compute $P(X=3)$.

When forming the second committee, there are $\binom{5}{3}$ ways to choose the members from the first committee and $\binom{10}{2}$ ways to choose the members who are not.

$P(X=3) = \frac{\binom{10}{2}\binom{5}{3}}{\binom{15}{5}} = \frac{150}{1001}$.

Problem 9

SKIP THIS PROBLEM ☐

Archie's Miscellaneous Supplies manufactures widgets. Each widget is, independently from the others, defective with probability .05; widgets are sold in cases of 1000. Let X be the number of defective widgets in a given case; let Y be the number of nondefective widgets in a given case.

X is binomial with $n = 1000, p = 0.05$; Y is binomial with $n = 1000, p = 0.95$.

a) Compute $E[X]$.

$1000(0.05) = 50$

b) Compute $E[Y]$.

$1000(0.95) = 950$

c) Compute $\mathrm{Var}[X]$.

$1000(0.05)(0.95) = 47.5$

d) Compute $\mathrm{Var}[Y]$.

$1000(0.95)(0.05) = 47.5$

Problem 10

SKIP THIS PROBLEM ☐

I have two urns, each of which contains 5 balls. I carry out the following four-step procedure.

- I transfer a ball at random from urn I to urn II.
- I transfer a ball at random from urn II to urn I.
- I transfer a ball at random from urn I to urn II.
- I transfer a ball at random from urn II to urn I.

What is the probability that I never move the same ball twice?

Let E_i be the event that the ball I move in step i is one I've never touched before.

$P(E_1E_2E_3E_4) = P(E_1E_2E_3E_4|E_1E_2E_3)P(E_1E_2E_3|E_1E_2)P(E_1E_2|E_1)P(E_1) = (\frac{4}{6})(\frac{4}{5})(\frac{5}{6})(1) = \frac{4}{9}$

Problem 11

SKIP THIS PROBLEM ☐

A certain large class has 100 students. During the semester, the students work on two projects in pairs. The students are divided up into 50 pairs randomly. The selection is done in such a way that on each project, all possible arrangements into 50 pairs are equally likely, and furthermore the pairings on two projects are independent of one another.

a) Two of the students in the class are named Alice and Betsy. Let E_{AB} be the event that Alice and Betsy are partners for the first project. Compute $P(E_{AB})$.

   All 99 of Alice's classmates are equally likely to be her partner for the first project. $P(E_{AB}) = \frac{1}{99}$.

b) Let X be the number of people who have the same partner for both projects. Compute $E[X]$.

   Let X_i be the random variable which is 1 if person i has the same partner twice and 0 otherwise, so that $X = \sum_{i=1}^{100} X_i$. Reasoning as in (a), each $E[X_i] = \frac{1}{99}$. So $E[X] = \frac{100}{99}$.