# Midterm (Sample Version 2, with Solutions)

## Math 425-201 Su10

by Prof. Michael "Cap" Khoury

| Name: | |
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## Directions:

Please print your name legibly in the box above.

You have 110 minutes to complete this exam. You may use any type of calculator without a QWERTY keyboard, but no computers/cell phones/iPads/etc. You may use two $3 \times 5$ notecards (front and back) but no other reference materials of any kind. There are eleven problems which are worth 16 points each. Partial credit is possible. *For full credit, clear and relevant work must be shown.*

You choose *ten* of the eleven problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, then whichever problem is skipped by most students becomes the default skip. This is not what you want.

This exam is worth 160 points, representing 40% of the 400 total points possible in the course.

This exam consists of 6 pages, front and back (one side for this cover page and one side for each of 11 problems). If you do not have all 6 pages or if any sides are blank, please notify me immediately!

| Problem Number | Value | Score |
|---|---|---|
| 1 | 16 | |
| 2 | 16 | |
| 3 | 16 | |
| 4 | 16 | |
| 5 | 16 | |
| 6 | 16 | |
| 7 | 16 | |
| 8 | 16 | |
| 9 | 16 | |
| 10 | 16 | |
| 11 | 16 | |
| **TOTAL** | **160** | |

## Problem 1

SKIP THIS PROBLEM ☐

Alice rolls two fair, independent six-sided dice. Let $X$ be the larger of the two numbers rolled, and let $Y$ be the smaller. (If both dice turn up the same number, then $X$ and $Y$ are both that common number.)

a) What are all possible values of $X$, and what is the probability that each will occur. (In other words, write down the pmf for $X$.

The possible values for $X$ are $1, 2, 3, 4, 5, 6$. The probabilities can be computed be simply listing cases.

$$\begin{aligned} P(X=1) &= 1/36 \\ P(X=2) &= 3/36 \\ P(X=3) &= 5/36 \\ P(X=4) &= 7/36 \\ P(X=5) &= 9/36 \\ P(X=6) &= 11/36 \end{aligned}$$

b) Compute $E[X]$.

$$\begin{aligned} E[X] &= 1(1/36)+2(3/36)+3(5/36)+4(7/36)+5(9/36)+6(11/36) \\ &= 161/36 \end{aligned}$$

c) Compute $\mathrm{Var}[X]$.

$$\begin{aligned} E[X^2] &= 1(1/36)+4(3/36)+9(5/36)+16(7/36)+25(9/36)+36(11/36) \\ &= 791/36 \\ \mathrm{Var}[X] &= 791/36-(161/36)^2 \end{aligned}$$

d) Compute $E[X+Y]$.

Since $X+Y$ is the sum of two fair dice, we already know its expectation is 7.

## Problem 2

SKIP THIS PROBLEM □

Suppose that $X, Y, Z$ are independent Poisson random variables with means 1, 2, and 3 (respectively).

a) What is the probability that the product $XYZ$ equals 1?

$P(XYZ=1) = P(X=1)P(Y=1)P(Z=1) = (1e^{-1})(2e^{-2})(3e^{-3}) = 6e^{-6}$.

b) What is the probability that the product $XYZ$ equals 0?

$$\begin{aligned} P(XYZ=0) &= 1 - P(XYZ \neq 0) \\ &= 1 - P(X \neq 0)P(Y \neq 0)P(Z \neq 0) \\ &= 1 - (1-e^{-1})(1-2e^{-2})(1-3e^{-3}) \end{aligned}$$

## Problem 3

SKIP THIS PROBLEM ☐

Alice and Betsy play a game in which each player flips 10 coins. (Assume that the 20 coins are independent and fair.) Alice gets one point for each "heads" among her 10 coins, and Betsy gets one point for each "tails" among her 10 coins. Let $X$ be Alice's score, and let $Y$ be Betsy's score.

a) What is $E[X]$?

$X$ is binomial with $n=10, p=1/2$, so $E[X]=5$.

b) Compute $P(X = Y)$, the probability of a tie. (Hint: think about how many total heads and tails occur when they tie.)

Note that $X = Y$ happens exactly when there are a total of 10 heads flipped on the 20 coins. This happens with probability $\frac{\binom{20}{10}}{2^{20}}$.

c) Compute $P(X > Y)$, the probability that Alice wins.

First, the probability that Alice *or* Betsy wins is $1 - \frac{\binom{20}{10}}{2^{20}}$; since Alice and Betsy are equally likely to win, $P(X > Y) = \frac{2^{20} - \binom{20}{10}}{2^{21}}$.

## Problem 4

.SKIP THIS PROBLEM ☐

I like to play a certain video game online with my friend Joe. Unfortunately, when we play, with probability 0.10 he will have connection problems on any given round. With probability 0.3, he will have no connection problems but will be talking to his wife while playing. With probability 0.6, he plays without distractions or connection problems.

In rounds where Joe has connection problems, I win with probability .95. In rounds where Joe is talking to his wife (but no connection problems), I win with probability .7. In rounds where Joe plays without distractions or connection problems, we are evenly matched, and I win with probability .5.

a) What is the probability that I win any individual round?

Let $A_1$ be the event that Joe has connection problems, $A_2$ the event that Joe is distracted, and $A_3$ the event that Joe is has no connection problems and is not distracted. Let $E$ be the event that I win.

$$\begin{aligned} P(E) &= P(E|A_1)P(A_1) + P(E|A_2)P(A_2) + P(E|A_3)P(A_3) \\ &= (.95)(.1) + (.7)(.3) + (.5)(.6) \\ &= 0.605 \end{aligned}$$

b) If I win the first round, what is the probability that Joe had connection problems on that round?

$$\begin{aligned} P(A_1|E) &= \frac{P(E|A_1)P(A_1)}{P(E)} \\ &= \frac{(0.95)(0.1)}{0.605} \\ &= \frac{19}{121} \end{aligned}$$

## Problem 5

SKIP THIS PROBLEM ☐

The variable $Z$ is a discrete random variable with a Zeta distribution ($\alpha = 3$). That is, $Z$ takes positive integer values, and $P(Z = n) = \frac{C}{n^4}$ for some suitable $C$.

In case you were (for any reason) wondering, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

a) Compute $P(Z = 1)$.

First, the given fact tells us that $C = \frac{90}{\pi^4}$.

$P(Z = 1) = \frac{C}{1} = \frac{90}{\pi^4}$.

b) Compute $P(Z = 3|Z \leqslant 3)$. Your answer for this part should not involve $\pi$ or decimal approximations.

$$\begin{aligned} P(Z = 3|Z \leqslant 3) &= \frac{P(Z = 3)}{P(Z = 1) + P(Z = 2) + P(Z = 3)} \\ &= \frac{C\frac{1}{81}}{C(1 + \frac{1}{16} + \frac{1}{81})} \\ &= \frac{16}{1393} \end{aligned}$$

## Problem 6

SKIP THIS PROBLEM ☐

You have ten novels, and you want to choose five to bring with you on a beach vacation. How many possible choices are there...

a) ...if two of the books are so similar that you don't want to bring both?

There are $\binom{10}{5} = 252$ choices if we don't care what books we bring. There are $\binom{8}{3} = 56$ which include both selected books, so there are 196 that don't include both of the books.

b) ...if three of the books are a trilogy, and you want to bring either none of the three or all three?

There are $\binom{7}{2} = 21$ ways to bring the trilogy and $\binom{7}{5} = 21$ to bring no books from the trilogy, for a total of 42.

## Problem 7

SKIP THIS PROBLEM ☐

Each of 15 people flips two (fair, independent) coins, a penny and a nickel. Everyone whose penny comes up heads forms the Penny Club. Everyone whose nickel comes up tails forms the Nickel Club. (It is possible that either or both clubs is empty.) Let $X$ be the number of people in the Penny Club. Let $Y$ be the number of people belonging to at least one club.

a) What is $E[X]$?

$X$ is binomial with $n = 15, p = 1/2$, so $E[X] = \frac{15}{2}$.

b) What is $E[Y]$?

Each person is in at least one club unless they flip $(\text{tails}, \text{heads})$. Thus $Y$ is binomial with $n = 15, p = 3/4$, $E[Y] = \frac{45}{4}$.

## Problem 8

SKIP THIS PROBLEM □

Henry (a man) and Liz (a woman) start a club. If at any time there are $m$ men and $w$ women in the club, the next new member of the club will be a man with probability $\frac{m}{m+w}$ and a woman with probability is $\frac{w}{m+w}$. (So, the third member is equally likely to be a man or a woman.)

a) Compute the probability that, when the club has ten members, Henry is the only man among them.

For $i = 3, 4, 5, \ldots 10$, let $M_i$ (resp. $W_i$) be the event that the $i$th member of the club is a man (resp. a woman).

Let $E_i = W_3 W_4 W_5 \cdots W_i$ be the event that all of the first $i$ people (except Henry) are women.

$$\begin{aligned} P(E_{10}) &= P(E_{10}|E_9)P(E_9|E_8)P(E_8|E_7)\cdots P(E_4|E_3)P(E_3) \\ &= \frac{8}{9}\cdot\frac{7}{8}\cdot\frac{6}{7}\cdot\frac{5}{6}\cdot\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2} \\ &= \frac{1}{9} \end{aligned}$$

b) Compute the probability that, when the club has five members, exactly three are women.

Let $E$ be the even that three of the five members are women.

$$\begin{aligned} P(E) &= P(W_3 W_4 M_5) + P(W_3 M_4 W_5) + P(M_3 W_4 W_5) \\ &= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{4} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{4} \\ &= \frac{1}{4} \end{aligned}$$

## Problem 9

SKIP THIS PROBLEM ☐

Imagine that I have a (fair) quadrillion-sided die, labelled with the numbers from 1 to one quadrillion. I roll this die two quadrillion times, each roll independent of all the others. Let $X$ be the number of times I roll a 19. (One quadrillion is an enormous number, so maybe you should imagine that I do all this via computer simulation.)

Give an approximation to $P(X \leqslant 2)$ which does not involve binomial coefficients.

First, notice that $X$ is binomial with $n$ large and $p$ small, so we can treat $X$ as a Poisson random variable with mean 2.

$$P(X \leqslant 2) \approx P(X=0) + P(X=1) + P(X=2) = \frac{1}{1}e^{-2} + \frac{2}{1}e^{-2} + \frac{4}{2}e^{-2} = \frac{5}{e^2}$$

## Problem 10

SKIP THIS PROBLEM ☐

I choose a positive integer $n$ at random, between 1 and 3600, each number being equally likely.

a) Compute $P(n \text{ is a square number})$.

There are 60 possible square numbers. The probability is $\frac{60}{3600} = \frac{1}{60}$.

b) Compute $P(\text{the fraction } \frac{n}{30} \text{ is in lowest terms})$.

Let $E_i$ be the event that $n$ is divisible by $i$ for $i = 2, 3, 5$.

$$\begin{aligned} 1 - P(E_2 \cup E_3 \cup E_5) &= 1 - P(E_2) - P(E_3) - P(E_5) + P(E_2E_3) + P(E_2E_5) + P(E_3E_5) \\ &\quad \cdots - P(E_2E_3E_5) \\ &= 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} - \frac{1}{30} \\ &= \frac{4}{15} \end{aligned}$$

## Problem 11

SKIP THIS PROBLEM ☐

An ordinary deck of 52 cards is shuffled.

a) What is the probability that the top 13 cards are all the same suit?

There are 52! possible arrangements of the deck.

There are 4 possible suits; for each suit, there are (13!)(39!) ways to arrange those cards on top and the other cards beneath.

The probability is $\frac{4(13!)(39!)}{52!}$

b) What is the probability that the top 13 cards contain at least 1 spade?

We compute first the probability that there are no spades in the top 13 cards. There are $\binom{39}{13}$ ways to choose the top 13 cards, 13! ways to sort them, and 39! ways to sort the other 39 cards underneath them.

The probability of no spades is $\frac{\binom{39}{13}(13!)(39!)}{52!}$.

The probability of at least one spade is $\frac{52! - \binom{39}{13}(13!)(39!)}{52!}$.

c) What is the probability that the spades are in their correct order from ace to king, not necessarily consecutively?

There are 13! equally-likely orders that the spades might be in.

The probability is $\frac{1}{13!}$.