Midterm (Sample Version 3, with Solutions)

Math 425-201 Su10

by Prof. Michael “Cap” Khoury

Name:	

Directions:

Please print your name legibly in the box above.

You have 110 minutes to complete this exam. You may use any type of calculator without a QWERTY keyboard, but no computers/cell phones/iPads/etc. You may use two 3×5 notecards (front and back) but no other reference materials of any kind. There are eleven problems which are worth 16 points each. Partial credit is possible. *For full credit, clear and relevant work must be shown.*

You choose *ten* of the eleven problems to complete. Indicate which problem you are skipping by marking the appropriate box in the upper-right hand corner of that page. If you do not mark any skip box, or if you mark more than one skip box, then whichever problem is skipped by most students becomes the default skip. This is not what you want.

This exam is worth 160 points, representing 40% of the 400 total points possible in the course.

This exam consists of 6 pages, front and back (one side for this cover page and one side for each of 11 problems). If you do not have all 6 pages or if any sides are blank, please notify me immediately!

Problem Number	Value	Score
1	16	
2	16	
3	16	
4	16	
5	16	
6	16	
7	16	
8	16	
9	16	
10	16	
11	16	
TOTAL	**160**	

Problem 1

SKIP THIS PROBLEM ☐

Morgan has six colored 12-sided dice: one red, one blue, one green, one purple, one orange, and one yellow. Each die, independently, turns up each of the numbers $1, 2, 3, \ldots, 12$ with equal probability. She rolls all six dice. Let R, O, Y, G, B, P be the numbers on the corresponding dice.

a) What is the probability that all the dice show different numbers?

Of the 12^6 rolls, $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$ have all numbers different.

The probability is $\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{12^6}$.

b) What is the probability that at least two of the dice show the same number?

The probability is $1 - \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{12^6} = \frac{12^6 - 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{12^6}$.

c) What is the probability that $R < O < Y < G < B < P$?

(Hint: if all the dice show different numbers, how many different orders might they be in?)

Given that the dice are all different totals, each of the 6! orderings is equally likely.

The probability is $\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{(6!)12^6}$.

Problem 2

SKIP THIS PROBLEM ☐

I have a collection of four coins. One comes up heads with probability .5, one comes up heads with probability .8, one comes up heads with probability .4, and one comes up heads with probability .3.

a) If I choose a coin at random and flip it, what is the conditional probability that I chose the fair coin, given that it came up heads?

Let E_i be the event that the ith coin is chosen (in the order listed above), and let H be the event that heads comes up.

$$\begin{aligned} P(E_1|H) &= \frac{P(H|E_1)P(E_1)}{\sum_{i=1}^4 P(H|E_i)P(E_i)} \\ &= \frac{(.5)(1/4)}{(.5)(1/4)+(.8)(1/4)+(.4)(1/4)+(.3)(1/4)} \\ &= \frac{1}{4} \end{aligned}$$

b) If I choose a coin at random and flip it five times, what is the conditional probability that I chose the fair coin, given that it came up heads exactly twice?

Let X be the number of heads flipped up. Notice that X is "conditionally binomial" given which of the E_i occurs.

$$\begin{aligned} P(X=2|H) &= \frac{P(X=2|E_1)P(E_1)}{\sum_{i=1}^4 P(X=2|E_i)P(E_i)} \\ &= \frac{\binom{5}{2}(.5)^2(.5)^3(1/4)}{\binom{5}{2}(1/4)((.5)^2(.5)^3+(.8)^2(.2)^3+(.4)^2(.6)^3+(.3)^2(.7)^3)} \end{aligned}$$

Problem 3

SKIP THIS PROBLEM □

How many ways are there to put identical markers on eight squares in an 8×8 grid so that each row and column has exactly one marker in it?

There are 8 ways to decide which square in the first row is marked, 7 ways to decide which square in the second row is marked, 6 ways for the third row, etc. There are $8!$ arrangements of markers.

Problem 4

SKIP THIS PROBLEM ☐

Mr. Brown is interviewing people for a job opening. At each interview, he asks the applicant the same three questions. If at least two of the applicant's answers impress him, Mr. Brown considers the applicant a good candidate for the job and invites her back for a second interview at a later time.

Mr. Brown is impressed by an answer to his first question with probability 0.7, to his second question with probability 0.5, and by an answer to his third question with probability 0.3. Whether an applicant's answer to one question is impressive is assumed to be independent of the other questions, and all applicants are assumed independent of one another.

Mr. Brown wants to have a total 10 candidates return for second interviews, so he stops interviewing applicants when he finds his 10th candidate. Let X be the number of people he interviews in order to find 10 candidates (assume he never runs out of applicants).

a) What is the probability that the first applicant becomes a candidate?

The probability is $(.7)(.5)(.3) + (.7)(.5)(.7) + (.7)(.5)(.3) + (.3)(.5)(.3) = .5$

b) Compute $E[X]$, the expected number of people interviewed.

X is negative binomial with $p = 1/2, r = 10$. $E[X] = 20$.

Problem 5

SKIP THIS PROBLEM □

There are n married couples, including my wife and me, seated around a round table with $2n$ seats. Seats have been assigned randomly, with all the possibilities equally likely. Let E_i be the event that couple i is seated together. Let X be the number of married couples who are seated together (so X takes values from 0 to n).

a) Compute $P(E_1)$, the probability that I am sitting next to my wife.

   I will be sitting next to 2 of the $2n-1$ other people, so $P(E_1) = \frac{2}{2n-1}$.

b) Compute $E[X]$, the expected number of couples seated together.

   Let X_i be 1 if E_i occurs and 0 otherwise. Then $E[X] = \sum E[X_i] = n\left(\frac{2}{2n-1}\right) = \frac{2n}{2n-1}$.

c) Assume that n is large enough, and the events E_i independent enough, that X is well approximated by a Poisson random variable. Approximate the probability that no couples are seated together.

   Let Y be Poisson with mean $\frac{2n}{2n-1}$. Then $P(X=0) \approx P(Y=0) = e^{-\frac{2n}{2n-1}}$.

Problem 6

SKIP THIS PROBLEM ☐

My daughter Datura is a big fan of the Disney princesses. Her two favorites are Ariel and Belle. Between each day and the next, she keeps the same favorite with probability 2/3 and switches (from Ariel to Belle or vice versa). Whether she changes her mind on a given day is independent of all the other days.

a) Two days before her birthday, Datura's favorite princess was Belle. I can order either an Ariel- or a Belle-themed birthday cake. Assuming I want to maximize the probability that the cake matches her favorite on her birthday, which cake should I order and how likely is it to be her favorite?

Let E_{AB} be the event that Datura prefers Ariel the day before her birthday and Belle on her birthday (and similarly for the other combinations).

$P(E_{BB}) + P(E_{AB}) = (2/3)(2/3) + (1/3)(1/3) = 5/9.$

I should order a Belle cake; I will be right with probability 5/9.

b) Given that Datura's favorite princess was Ariel two days ago and that Datura's favorite princess is Ariel today, what is the conditional probability that her favorite princess was Ariel yesterday?

Let A_1 (resp. B_1) be the event that Datura prefers Ariel (resp. Belle) yesterday, and let A_2, B_2 be the corresponding things for today. (We take Datura's preference two days ago as part of all our probabilities.

$$\begin{aligned} P(A_1|A_2) &= \frac{P(A_1A_2)}{P(B_1A_2) + P(A_1A_2)} \\ &= \frac{4/9}{1/9 + 4/9} \\ &= \frac{4}{5} \end{aligned}$$

Problem 7

SKIP THIS PROBLEM ☐

An ordinary 52-card deck is thoroughly shuffled. You draw cards, one at a time, from the top of the deck until the first time you have a pair of cards of the same rank, then stop.

a) What is the probability that you draw exactly ten cards?

Work card-by-card. The probability is $\frac{52}{52} \cdot \frac{48}{51} \cdot \frac{44}{50} \cdot \frac{40}{49} \cdot \frac{36}{48} \cdot \frac{32}{47} \cdot \frac{28}{46} \cdot \frac{24}{45} \cdot \frac{20}{44} \cdot \frac{27}{43}$

b) What is the probability that the pair you get is a pair of jacks?

Each of the 13 ranks is equally likely; the probability is $\frac{1}{13}$.

Problem 8

SKIP THIS PROBLEM ☐

The card game of "Robot Builder" is played with a special deck of 60 cards, including 20 different "head" cards, 20 different "torso" cards, and 20 different "legs" cards. At the beginning of the game, the deck is shuffled and each player draws seven cards.

a) How many different starting hands are possible? (As noted above, all the cards are different from one another.)

$\binom{60}{7}$

b) The goal of the game is to build robots by combining a head with a torso and legs. What is the probability that your starting hand lets you build two robots immediately? (That is, what is the probability that your starting hand contains at least two heads, at least two torsos, and at least two legs.)

Your hand will have either two heads, two torsos, and three legs; two heads, three torsos, and two legs; or three heads, two torsos, and two legs. In each case, the number of possible hands is $\binom{20}{2}\binom{20}{2}\binom{20}{3}$, for a total of $3\binom{20}{2}\binom{20}{2}\binom{20}{3}$.

The probability is $\frac{3\binom{20}{2}\binom{20}{2}\binom{20}{3}}{\binom{60}{7}}$.

Problem 9

SKIP THIS PROBLEM □

Salty Jack is a pirate captain, and he swaggers into Sketchy Pete's tavern, looking for pirates to fill up his crew. After he gives his best recruitment speech, there are 20 people in the tavern interested in joining the crew. Suppose that the probability of any interested person actually being capable of doing pirate's work is $3/5$, independently of any other people. Let X be the number of capable people willing to join the crew. (So X takes the possible values $0, 1, 2, \ldots, 20$.)

It turns out that Jack is quite shorthanded and *needs* at least three people to join his crew; he'll take even unqualified people if necessary to get the total up to 3. Let Y be the number of new pirates Salty Jack hires. (So Y takes the possible values $3, 4, 5, \ldots, 20$.)

a) What is $E[X]$?

The random variable X is binomial with $n = 20, p = 3/5$. $E[X] = 12$.

b) What is $\mathrm{Var}[X]$?

$\mathrm{Var}[X] = 20(3/5)(2/5) = 4.8$

c) Compute $P[Y = 3]$.

$$\begin{aligned}
P[Y=3] &= P[X=0] + P[X=1] + P[X=2] + P[X=3] \\
&= \binom{20}{0}\left(\frac{2}{5}\right)^0\left(\frac{3}{5}\right)^{20} + \binom{20}{1}\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^{19} + \binom{20}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^{18} + \\
&\quad \binom{20}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^{17}
\end{aligned}$$

Problem 10

SKIP THIS PROBLEM □

The discrete random variable X takes on positive integer values. The cdf of X is as follows.

$$F(n) = P(X \leqslant n) = \frac{n^2 - 1}{n^2 + 1} \qquad n = 1, 2, 3, \ldots$$

a) Compute $P(X = 3)$.

$P(X = 3) = F(3) - F(2) = \frac{8}{10} - \frac{3}{5} = \frac{1}{5}$

b) Compute $P(X = 3 | X \leqslant 3)$.

$P(X = 3 | X \leqslant 3) = \frac{P(X = 3)}{F(3)} = \frac{1/5}{8/10} = \frac{1}{4}$

c) Compute $P(X = 3 | X \geqslant 3)$.

$P(X = 3 | X \geqslant 3) = \frac{P(X = 3)}{1 - F(2)} = \frac{1/5}{1 - 3/5} = \frac{1}{2}$

Problem 11

SKIP THIS PROBLEM ☐

Alice and Betsy are playing a game with two dice. Alice and Betsy each choose a different number between 1 and 6. (So, for example, Alice might choose 2 and Betsy might choose 5.) They roll a pair of dice. If Alice's number comes up on either die and Betsy's doesn't, Alice wins. If Betsy's number comes up on either die and Alice's doesn't, Betsy wins. If neither number comes up, or if both do, they roll again. Let X be the number of rolls it takes to determine a winner.

a) Compute $P(\text{Alice wins})$.

Alice and Betsy are equally likely to win, so each will win with probability 1/2.

b) Compute $E[X]$, the expected number of rolls needed to determine a winner.

The probability that any given roll is the last one is as follows.

$$1 - P(\text{neither number}) - P(\text{both numbers}) = 1 - \frac{16}{36} - \frac{2}{36} = \frac{1}{2}$$

Thus X is geometric with $p = 1/2$, so $E[X] = 2$.