Answers for Exam 1 Example A

1. By the inclusion–exclusion principle, the proportion is

$$1 - (.2 + .15 + .1) + (.05 + .04 + .03) - (.01) = 0.66$$

2. First we look for solutions of the form $u_n = \lambda^n$. Such a λ would satisfy $\lambda^2 - \lambda - 6 = 0$. Here the left hand side factors as $(\lambda-3)(\lambda+2)$, so $\lambda = 3$ or $\lambda = -2$. Thus $u_n = \alpha 3^n + \beta(-2)^n$, for some choice of α and β. To determine the values of these constants, we use the initial conditions: $u_0 = 2 = \alpha + \beta$, and $u_1 = 3\alpha - 2\beta$. By elimination we find that $\alpha = \beta = 1$. That is, $u_n = 3^n + (-2)^n$.

3. (a)

$$\binom{52}{13\ 13\ 13\ 13} = \frac{52!}{13!^4}$$

(b) The aces can be distributed in $\binom{4}{1\ 1\ 1\ 1} = 4! = 24$ ways. The non-aces can be distributed in $\binom{48}{12\ 12\ 12\ 12}$ ways. Hence the answer is $\binom{4}{1\ 1\ 1\ 1}\binom{48}{12\ 12\ 12\ 12} = 4!48!/12!^4$.

(c)

$$\frac{4!48!/12!^4}{52!/13!^4} = \frac{24 \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{13^3}{17 \cdot 25 \cdot 49} = 0.105498\,.$$

4. 1. $0 \leq P(E) \leq 1$ for all events $E \subseteq S$. 2. $P(S) = 1$. 3. If $E_1, E_2, E_3, \ldots$ are pairwise mutually exclusive events, then

$$P\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} P(E_i)\,.$$

5. (a) Let F be the event that the fair coin was chosen, and H_1 the event that it comes up heads. By conditioning, we see that $P(H_1) = P(H_1|F)P(F) + P(H_1|F^c)P(F^c) = (1/2)(1/2) + 1(1/2) = 3/4$.

(b) $P(F|H_1) = P(H_1|F)P(F)/P(H_1) = (1/2)(1/2)/(3/4) = 1/3$.

(c)

$$\begin{aligned} P(F|H_1H_2) &= \frac{P(H_1H_2|F)P(F)}{P(H_1H_2)} \\ &= \frac{(1/4)(1/2)}{P(H_1H_2|F)P(F) + P(H_1H_2|F^c)P(F^c)} \\ &= \frac{1/8}{(1/4)(1/2) + 1(1/2)} = \frac{1}{5} = 0.2\,. \end{aligned}$$

(d) 1, because the other coin never comes up tails.

6. (a) 1/13, because there are 52 cards, any one of which is equally likely to be in the fifth place, and there are 4 aces. $4/52 = 1/13$.
(b) Two solutions. First, by conditioning, we find that

$$\begin{aligned} P(N_1N_2N_3N_4A_1) &= P(N_1)P(N_2|N_1)P(N_3|N_1N_2)P(N_4|N_1N_2N_3)P(A_1|N_1N_2N_3N_4) \\ &= \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50}\cdot\frac{45}{49}\cdot\frac{4}{48} = \frac{3\cdot 23\cdot 47}{5\cdot 7^2\cdot 13\cdot 17} = 0.0598947\,. \end{aligned}$$

Second solution, by counting: Of the 47 places from the sixth through the 52nd, we choose three locations for the remaining aces to fall, in $\binom{47}{3}$ ways. We order the aces, in 4! ways, and we order the non-aces in 48! ways. Hence the answer is

$$\frac{\binom{47}{3}4!48!}{52!} = \frac{47\cdot 46\cdot 45\cdot 24}{6\cdot 52\cdot 51\cdot 50\cdot 49} = \frac{3\cdot 23\cdot 47}{5\cdot 7^2\cdot 13\cdot 17} = 0.0598947\,.$$

(c) Let F denote the event that the fifth card is the first ace, and N the event that the next card is the ace of spades. We compute $P(N|F)$ in two ways. First we condition on whether the first ace is the ace of spades. Let A denote this event. Then

$$\begin{aligned} P(N|F) &= P(N|AF)P(A|F) + P(N|A^cF)P(A^c|F) \\ &= 0\cdot P(A|F) + \frac{1}{47}\cdot\frac{3}{4} = \frac{3}{2^2\cdot 47} = \frac{207}{216580} = 0.015957\,. \end{aligned}$$

Alternatively, we appeal to the identity $P(N|F) = P(NF)/P(F)$. We have already computed $P(F)$, and we find $P(NF)$ by counting: Of the 46 locations from the seventh through the 52nd, we choose 2, where the remaining aces fall, in $\binom{46}{2}$ ways. We know where the ace of spades falls. We order the remaining aces in 3! ways, and order the 48 non-aces in 48! ways. Hence

$$P(NF) = \frac{\binom{46}{2}3!48!}{52!} = \frac{46\cdot 45\cdot 6}{2\cdot 52\cdot 51\cdot 50\cdot 49} = \frac{3^2\cdot 23}{2^2\cdot 5\cdot 7^2\cdot 13\cdot 17} = \frac{207}{216580}$$

Hence the desired probability is

$$\frac{3}{2^2\cdot 47} = \frac{207}{216580} = 0.015957\,.$$