Answers for Exam 1 Example A

1. By the inclusion–exclusion principle, the proportion is

\[
1 - (0.2 + 0.15 + 0.1) + (0.05 + 0.04 + 0.03) - (0.01) = 0.66
\]

2. First we look for solutions of the form \( u_n = \lambda^n \). Such a \( \lambda \) would satisfy \( \lambda^2 - \lambda - 6 = 0 \). Here the left hand side factors as \((\lambda - 3)(\lambda + 2)\), so \( \lambda = 3 \) or \( \lambda = -2 \). Thus \( u_n = \alpha 3^n + \beta (-2)^n \), for some choice of \( \alpha \) and \( \beta \). To determine the values of these constants, we use the initial conditions: \( u_0 = 2 = \alpha + \beta \), and \( u_1 = 3\alpha - 2\beta \). By elimination we find that \( \alpha = \beta = 1 \). That is, \( u_n = 3^n + (-2)^n \).

3. (a)

\[
\binom{52}{13, 13, 13, 13} = \frac{52!}{13!^4}
\]

(b) The aces can be distributed in \( \binom{4}{1, 1, 1, 1} = 4! = 24 \) ways. The non-aces can be distributed in \( \binom{48}{12, 12, 12, 12} \) ways. Hence the answer is \( \binom{4}{1, 1, 1, 1} \binom{48}{12, 12, 12, 12} = \frac{4!48!}{12!^4} \).

(c) \[
\frac{4!48!}{52!} = \frac{24 \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{13^4}{17 \cdot 25 \cdot 49} = 0.105498.
\]

4. 1. \( 0 \leq P(E) \leq 1 \) for all events \( E \subseteq S \). 2. \( P(S) = 1 \). 3. If \( E_1, E_2, E_3, \ldots \) are pairwise mutually exclusive events, then

\[
P\left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} P(E_i).
\]

5. (a) Let \( F \) be the event that the fair coin was chosen, and \( H_1 \) the event that it comes up heads. By conditioning, we see that \( P(H_1) = P(H_1|F)P(F) + P(H_1|F^c)P(F^c) = (1/2)(1/2) + 1(1/2) = 3/4 \).

(b) \( P(F|H_1) = P(H_1|F)P(F)/P(H_1) = (1/2)(1/2)/(3/4) = 1/3 \).

(c) \[
P(F|H_1H_2) = \frac{P(H_1H_2|F)P(F)}{P(H_1H_2)}
= \frac{(1/4)(1/2)}{P(H_1H_2|F)P(F) + P(H_1H_2|F^c)P(F^c)}
= \frac{1/8}{(1/4)(1/2) + 1(1/2)} = \frac{1}{5} = 0.2.
\]

(d) 1, because the other coin never comes up tails.
6. (a) 1/13, because there are 52 cards, any one of which is equally likely to be in
the fifth place, and there are 4 aces. 4/52 = 1/13.
(b) Two solutions. First, by conditioning, we find that

\[
P(N_1N_2N_3N_4A_1) = P(N_1)P(N_2|N_1)P(N_3|N_1N_2)P(N_4|N_1N_2N_3)P(A_1|N_1N_2N_3N_4)
\]

\[
= \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \cdot \frac{4}{48} = \frac{3 \cdot 23 \cdot 47}{5 \cdot 7^2 \cdot 13 \cdot 17} = 0.0598947
\]

Second solution, by counting: Of the 47 places from the sixth through the 52nd,
we choose three locations for the remaining aces to fall, in \( \binom{47}{3} \) ways. We order
the aces, in 4! ways, and we order the non-aces in 48! ways. Hence the answer is

\[
\frac{\binom{47}{3} \cdot 4! \cdot 48!}{52!} = \frac{47 \cdot 46 \cdot 45 \cdot 24}{6 \cdot 52 \cdot 51 \cdot 50 \cdot 49} = \frac{3 \cdot 23 \cdot 47}{5 \cdot 7^2 \cdot 13 \cdot 17} = 0.0598947
\]

(c) Let \( F \) denote the event that the fifth card is the first ace, and \( N \) the event that
the next card is the ace of spades. We compute \( P(N|F) \) in two ways. First we
condition on whether the first ace is the ace of spades. Let \( A \) denote this event.
Then

\[
P(N|F) = P(N|AF)P(A|F) + P(N|A^cF)P(A^c|F)
\]

\[
= 0 \cdot P(A|F) + 1 \cdot \frac{3}{47} \cdot \frac{3}{4} = \frac{3}{2^2 \cdot 47} = \frac{207}{216580} = 0.015957
\]

Alternatively, we appeal to the identity \( P(N|F) = P(NF)/P(F) \). We have already
computed \( P(F) \), and we find \( P(NF) \) by counting: Of the 46 locations from the
seventh through the 52nd, we choose 2, where the remaining aces fall, in \( \binom{46}{2} \) ways.
We know where the ace of spades falls. We order the remaining aces in 3! ways, and
order the 48 non-aces in 48! ways. Hence

\[
P(NF) = \binom{46}{2} \cdot 3! \cdot 48! = \frac{46 \cdot 45 \cdot 6}{2 \cdot 52 \cdot 51 \cdot 50 \cdot 49} = \frac{3^2 \cdot 23}{2^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17} = \frac{207}{216580}
\]

Hence the desired probability is

\[
\frac{3}{2^2 \cdot 47} = \frac{207}{216580} = 0.015957
\]