1. Let \( X \) denote a random variable with \( E[X] = 0 \) and \( \text{Var}(X) = 1 \). Put \( Y = 2X + 3 \). (a) \( E[Y] = 2E[X] + 3 = 5 \). (b) \( \text{Var}(Y) = 2^2\text{Var}(X) = 4 \). (c) Assume now that \( X \) is a normal variable. \( P(Y \leq 4) = P(2X + 3 \leq 4) = P(2X \leq 1) = P(X \leq 1/2) = \Phi(.5) = 0.6915 \).

2. Let \( X \) denote a geometric random variable with parameter \( p \). (a) \( p_X(k) = (1 - p)^{k-1}p \) for \( k = 1, 2, 3, \ldots \). (b) \( E[X] = 1/p \). (c) Let \( a \) be a positive integer. \( P(X > a) = \sum_{k=a+1}^\infty (1-p)^{k-1}p = p(1-p)^a \sum_{r=0}^\infty (1-p)^r = p(1-p)^a/(1 - (1-p)) = (1-p)^a \). (d) Let \( a \) and \( b \) be positive integers. Then \( P(X > a+b|X > a) = P(X > a+b)/P(X > a) = (1-p)^{a+b}/(1-p)^a = (1-p)^b = P(X > b) \).

3. (a) A coin that comes up heads with probability \( p \) is repeatedly flipped, in independent trials, until it comes up head for the first time. Let \( X \) denote the number of flips. Then \( X \) is a geometric random variable with parameter \( p \). \( E[X] = 1/p \), \( \text{Var}(X) = (1-p)/p^2 \). (b) The same coin as above is flipped, in independent trials, until it comes up head for the \( r \)th time. Let \( Y \) denote the number of flips. Then \( Y \) is a negative binomial random variable with parameters \( r \) and \( p \). \( E[Y] = r/p \) and \( \text{Var}(Y) = r(1-p)/p^2 \). (c) A shop has on average 6 customers per hour. Let \( X \) denote the number of customers in a particular hour. Then \( X \) is a Poisson random variable with parameter \( \lambda = 6 \). \( E[X] = \lambda \) and \( \text{Var}(X) = \lambda \). (d) In the same shop, let \( T \) denote the elapsed time between 10:17am and the moment when the next customer enters the shop. Then \( T \) is an exponential random variable with parameter \( \lambda = 6 \). \( E[T] = 1/\lambda = 1/6 \) and \( \text{Var}(T) = 1/\lambda^2 = 1/36 \). (e) Suppose that there are 20 passengers on a bus, and that a passenger is male with probability 0.6, independently from passenger to passenger. Let \( X \) denote the number of male passengers on the bus. Then \( X \) is a binomial random variable with parameters \( n = 20 \) and \( p = 0.6 \). \( E[X] = np = 12 \) and \( \text{Var}(X) = np(1-p) = 4.8 \). (f) Suppose that there are 110,000 football fans in the Big House, and that a fan is male with probability 0.6, independently from fan to fan. Let \( X \) denote the number of male fans in the Big House. Then the quantity \( P(X > 66,100) \) would be estimated by using a normal random variable with parameters \( \mu = 66,000 \) and \( \sigma = \sqrt{np(1-p)} = \sqrt{26400} = 162.48 \).

4. A fire station is to be built along a road stretching from \((0, 0)\) to \((L, 0)\) in the coordinate plane. Let \((a, 0)\) denote the position of the fire station, where \(0 \leq a \leq L\). Assume that fires occur along the road at positions that are uniformly distributed from 0 to \(L\). Let \( X \) denote the distance that the fire truck must drive to reach a fire. (a) Let \((U, 0)\) denote the position of the fire. Then \( X = |U - a| \), so

\[
E[X] = \frac{1}{L} \int_0^L |u - a| \, du = \frac{1}{L} \int_0^a a - u \, du + \frac{1}{L} \int_a^L u - a \, du
= \frac{a^2}{2L} + \frac{(L-a)^2}{2L} = \frac{2a^2 - 2aL + L^2}{2L}.
\]

(b) The derivative of this with respect to \( a \) is \((4a-2L)/(2L) = (2a-L)/L\). This is negative when \(0 \leq a < L/2\) and positive when \(L/2 < a \leq L\), so the minimum of \(E[X]\) occurs when \(a = L/2\).
5. The random variables $X$ and $Y$ take the values 0 and 1 with joint distribution $p_{X,Y}(x,y)$ as given below.

\[
\begin{array}{ccc}
Y & 1/3 & 1/3 \\
0 & 1/6 & 1/6 \\
0 & 1 \\
X
\end{array}
\]

(a) $p_X(0) = 1/2$, $p_X(1) = 1/2$.  (b) $p_Y(0) = 1/3$, $p_Y(1) = 2/3$.  (c) $X$ and $Y$ are independent because $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for $x = 0,1$ and $y = 0,1$. 