Brief Answers to the Second Sample Second Exam

1. Let X denote a random variable with $E[X] = 0$ and $\mathrm{Var}(X) = 1$. Put $Y = 2X + 3$. (a) $E[Y] = 2E[X] + 3 = 5$. (b) $\mathrm{Var}(Y) = 2^2\mathrm{Var}(X) = 4$. (c) Assume now that X is a normal variable. $P(Y \le 4) = P(2X + 3 \le 4) = P(2X \le 1) = P(X \le 1/2) = \Phi(.5) = 0.6915$.

2. Let X denote a geometric random variable with parameter p. (a) $p_X(k) = (1-p)^{k-1}p$ for $k = 1, 2, 3, \ldots$. (b) $E[X] = 1/p$. (c) Let a be a positive integer. $P(X > a) = \sum_{k=a+1}^{\infty}(1-p)^{k-1}p = p(1-p)^a \sum_{r=0}^{\infty}(1-p)^r = p(1-p)^a/(1-(1-p)) = (1-p)^a$. (d) Let a and b be positive integers. Then $P(X > a + b | X > a) = P(X > a + b)/P(X > a) = (1-p)^{a+b}/(1-p)^a = (1-p)^b = P(X > b)$.

3. (a) A coin that comes up heads with probability p is repeatedly flipped, in independent trials, until it comes up head for the first time. Let X denote the number of flips. Then X is a geometric random variable with parameter p. $E[X] = 1/p$, $\mathrm{Var}(X) = (1-p)/p^2$. (b) The same coin as above is flipped, in independent trials, until it comes up head for the r^{th} time. Let Y denote the number of flips. Then Y is a negative binomial random variable with parameters r and p. $E[Y] = r/p$ and $\mathrm{Var}(Y) = r(1-p)/p^2$ (c) A shop has on average 6 customers per hour. Let X denote the number of customers in a particular hour. Then X is a Poisson random variable with parameter $\lambda = 6$. $E[X] = \lambda$ and $\mathrm{Var}(X) = \lambda$ (d) In the same shop, let T denote the elapsed time between 10:17am and the moment when the next customer enters the shop. Then T is an exponential random variable with parameter $\lambda = 6$. $E[T] = 1/\lambda = 1/6$ and $\mathrm{Var}(T) = 1/\lambda^2 = 1/36$. (e) Suppose that there are 20 passengers on a bus, and that a passenger is male with probability 0.6, independently from passenger to passenger. Let X denote the number of male passengers on the bus. Then X is a binomial random variable with parameters $n = 20$ and $p = 0.6$. $E[X] = np = 12$ and $\mathrm{Var}(X) = np(1-p) = 4.8$. (f) Suppose that there are 110,000 football fans in the Big House, and that a fan is male with probability 0.6, independently from fan to fan. Let X denote the number of male fans in the Big House. Then the quantity $P(X > 66{,}100)$ would be estimated by using a normal random variable with parameters $\mu = 66{,}000$ and $\sigma = \sqrt{np(1-p)} = \sqrt{26400} = 162.48$.

4. A fire station is to be built along a road stretching from $(0, 0)$ to $(L, 0)$ in the coordinate plane. Let $(a, 0)$ denote the position of the fire station, where $0 \le a \le L$. Assume that fires occur along the road at positions that are uniformly distributed from 0 to L. Let X denote the distance that the fire truck must drive to reach a fire. (a) Let $(U, 0)$ denote the position of the fire. Then $X = |U - a|$, so

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\begin{aligned}
E[X] &= \frac{1}{L}\int_0^L |u-a|\,du = \frac{1}{L}\int_0^a a - u\,du + \frac{1}{L}\int_a^L u - a\,du \\
&= \frac{a^2}{2L} + \frac{(L-a)^2}{2L} = \frac{2a^2 - 2aL + L^2}{2L}.
\end{aligned}
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(b) The derivative of this with respect to a is $(4a - 2L)/(2L) = (2a - L)/L$. This is negative when $0 \le a < L/2$ and positive when $L/2 < a \le L$, so the minimum of $E[X]$ occurs when $a = L/2$.

5. The random variables X and Y take the values 0 and 1 with joint distribution $p_{X,Y}(x,y)$ as given below.

Y	1	1/3	1/3
	0	1/6	1/6
		0	1
		X	

(a) $p_X(0) = 1/2$, $p_X(1) = 1/2$. (b) $p_Y(0) = 1/3$, $p_Y(1) = 2/3$. (c) X and Y are independent because $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for $x = 0, 1$ and $y = 0, 1$.