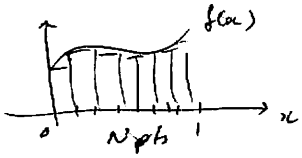
Lec 39 | 04/15

APPLICATIONS

1. Scientific computing : Monte-Carlo integration.

$\int_0^1 f(x)\,dx = ?$ (numerically)

- Riemann sums

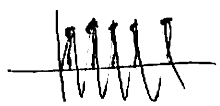


$\int f(x)\,dx \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i)$

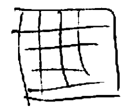
N pts x_i

- Difficulties:

(a). Oscillations



(b) High dimensions n: complexity exponential in n.



$\int_{[0,1]^n} f(x)\,dx$ $\qquad N^n$ points.

$10^{80} >$ # of atoms in the universe.

- Monte-Carlo (Manhattan Project, S. Ulam).

"Pick x_i at random".

$X,\ X_i \sim \mathrm{Unif}(0,1)$ indep.

$E[X] = \int_0^1 f(x)\,dx$

LLN: $\frac{1}{N}\sum_{i=1}^{N} f(X_i) \to E[X]$ as $N \to \infty$ (w/prob 1)

good estimate for $\int_0^1 f(x)\,dx$!

~~Advantage?~~

Rate of convergence: $O(\frac{1}{\sqrt{N}})$, ~~indep~~ depends only on the variance $\left(\int_0^1 f(x)^2 dx\right)$

but not on oscillation.

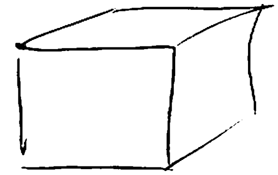
Equally good in any dimension n. $\left(X_i \sim \mathrm{Unif}([0,1]^n)\right)$.

2. Geometry: volume paradox in high dimensions.

CUBE $[0,1]^n$

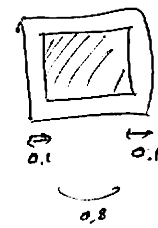
$X \sim \mathrm{Unif}([0,1]^n)$. Where is X most likely?

$\Leftrightarrow$ where is most of the volume of the cube?



(a) Near the surface.

$$X = (X_1, \dots, X_n), \qquad X_i \sim \mathrm{Unif}(0,1)$$



~~"Interior" part:~~

$\mathbb{P}\{X \in \text{interior part}\} = \mathbb{P}\{\text{all } 0.1 \le X_i \le 0.9\} = (0.8)^n$ by indep

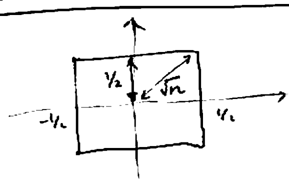
$\to 0$ as $n \to \infty$

$(0.8^{10} \approx 0.11)$

$\Rightarrow$ Difficult to get inside the cube in high dim!

Poses difficulties in applications.

(b) Near the corners.



Translate. $X \sim \mathrm{Unif}([-1/2, 1/2]^n)$

$$\mathbb{E}\,\|X\|^2 = \mathbb{E}\Big(\sum_1^n X_i^2\Big) = \sum_1^n \underbrace{\mathbb{E}X_i^2}_{1/12} = n/12$$

LLN: $\|X\|^2 \sim \frac{n}{12}$ whp

~~Moreover, LLN $\Rightarrow$~~

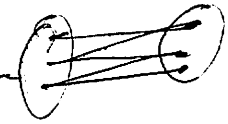
$$\tfrac{1}{n}\mathbb{E}\|X\|^2 = \mathbb{E}\Big[\frac{1}{n}\sum_1^n X_i^2\Big] = \frac{1}{12}$$

LLN $\Rightarrow$ $\dfrac{\|X\|^2}{n} \to \dfrac{1}{12}$.

$\Rightarrow$ $\|X\| \approx \sqrt{\frac{n}{12}}$ whp. Near corners!

3. Combinatorics (Alon-Spencer Thm 2.2.1)

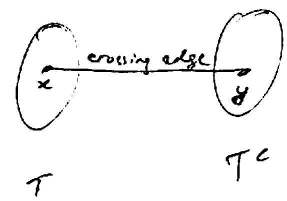
Thm Let G be a graph with k edges.
Then G contains a bipartite subgraph with $\geq \frac{k}{2}$ edges



* (There exists two group of vertices ~~s.t. ... so that half of the~~ ~~with~~ that are well connected)

• Probabilistic method:

Split G at random into two equal parts T, T^c
$\mathbb{P}\{x \in T\} = 1/2$ for every vertex indep.



Call edge (x,y) if it connects T to T^c.

The set of crossing edges forms a bipartite graph.

$X = \#$ of crossing edges

$$X = \sum_{\substack{(x,y) \in \\ \text{all edges of } G}} X_{xy}, \qquad X_{xy} = \begin{cases} 1, & (x,y) \text{ crossing} \\ 0, & \text{———} \end{cases}$$

$$E[X_{xy}] = P\{(x,y) \text{ crossing}\} = \frac{1}{2}$$ (two fair coins have prob $\frac{1}{2}$ of being different)

$$\Rightarrow \quad E[X] = \sum_{(x,y)} \frac{1}{2} = \frac{k}{2}.$$

~~Facts~~