· Previous class :

- Multiplication Rule
- Permutations : # of ways to arrange n objects in a line $= n! = n(n-1)\cdots 1$
- Combinations : # of ways to select k unordered objects out of n

(*) $$= \binom{n}{k} = \frac{n!}{k!(n-k)!} = \text{"Binomial coefficient", "n choose k"}.$$

Convention: $0! = 1$, so $\binom{n}{0} = \binom{n}{n} = 1$.

1.4. Combinations (contd)

Ex How many words can be made of 5 letters A, 3 letters B?

Ans $= \binom{8}{5} = 56$ (by choosing 5 places for A : _ A _ A A _ A A)

$= \binom{8}{3}$ (by choosing 3 places for B : B _ B _ _ B _ _)

More generally, we have

$$\boxed{\textbf{Proposition} \quad \binom{n}{k} = \binom{n}{n-k}}$$ — follows easily from (*).

Binomial Theorem For $x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Partial cases : $(x+y)^2 = x^2 + 2xy + y^2$ (note : $1 = \binom{2}{0}$, $2 = \binom{2}{1}$).

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ (note : $3 = \binom{3}{1} = \binom{3}{2}$).

Proof of Binomial Theorem.

Expand $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n} = \sum_{k=0}^{n} \underbrace{a_{n,k}}_{=\#\text{ of such monomials}} x^k y^{n-k}$

monomials like $\underbrace{xyxxyyxy\cdots x}_{n}$ where x appears k times

$a_{n,k} = \#$ of words that can be made of k letters x, $n-k$ letters y

$= \binom{n}{k}$.

QED.

Corollary $\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$ (by choosing $x=y=1$).

Interpretation of Corollary:

TIME PER-MITTING

Ex n courses are offered.
\# of ways to register?

Ans $= \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n} = 2^n$ by the multiplication principle
(for each course, I have two possibilities - register or not)

Remark $2^n = $ \# of subsets of an n-element set.

Remark Corollary above: two ways of counting:
RHS = \# of ways to register
LHS = \# of ways to register for exactly k courses, then sum over $k=0,\ldots n$.
obviously equal.

Ex (Lottery). "Pick 6 out of 49 correctly" $\Rightarrow$ win.

Probability to win $= \frac{1}{\binom{49}{6}} \approx 7.2 \cdot 10^{-8}$

Compare: get 23 heads in a row with a fair coin:

Probability $= 2^{-23} \approx 1.2 \cdot 10^{-7}$ $\to$ larger!

Ex (for home):
Probability that 5 numbers are guessed correctly, 1 incorrect
$= \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}} \approx 1.8 \cdot 10^{-5}$

Ex A lab has 10 rats, of which 4 are believed to be sick (40%).
5 rats are selected at random for a study.
What is the probability that the sample is representative, i.e. it contains 2 sick rats (still 40%)

\# of ways to select a sample $= \binom{10}{5} = 252$

\# of ways to select a sample with 2 sick rats $= \binom{4}{2}\binom{6}{3} = 120$.

2 sick out of 4 sick

3 healthy out of 6 healthy

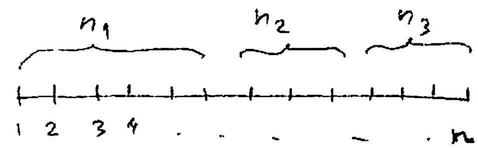
Probability $= \frac{120}{252} = 0.48$

1.5. Multinomial Coefficients.

Ex | # of ways to divide n employees into 3 teams of sizes n_1, n_2, n_3 (where of course $n = n_1 + n_2 + n_3$)

Remark: If there were two teams, n_1 and n_2, then $\text{Ans} = \binom{n}{n_1} = \dfrac{n!}{n_1!\, n_2!}$

Model: 1) Arrange the employees in a line



2) Assign the first n_1 for team 1,
next n_2 for team 2,
last n_3 for team 3

# of ways to do this is $n!$

But we have to ignore the order within the teams:

$$\Rightarrow \boxed{\text{Ans} = \frac{n!}{n_1!\, n_2!\, n_3!}} =: \binom{n}{n_1\ n_2\ n_3}.$$

Def (Multinomial coefficient) Let $n = n_1 + n_2 + \dots + n_r$

$$\binom{n}{n_1, n_2, \dots, n_r} := \frac{n!}{n_1!\, n_2! \dots n_r!}.$$

This equals # of ways to divide n objects into r $\overset{\text{distinct}}{\text{groups}}$ of sizes $n_1, n_2, \dots n_r$.

Ex | # of different letter arrangements of word "STATISTICS"?

There are 10 letters: 3S, 3T, 1A, 2I, 1C.

Model: divide 10 positions into 5 groups of sizes 3, 3, 1, 2, 1 (group "S", group "T", etc).

$$\text{\# of ways} = \binom{10}{3, 3, 1, 2, 1} = \frac{10!}{3!\,3!\,2!} = 50{,}400.$$

Ex: What is the probability that a child playing with these letters arranges them into the word "STATISTICS" at random?

$$\text{Ans} = \frac{1}{50{,}400} \approx 2 \cdot 10^{-5}$$

Remark on non-distinct groups → divide by $r!$ — see Example 5c