2.3. Axioms of probability.

S: sample space

Def A probability is an assignment of a number $P(E)$ to each event E, which satisfies the following three axioms:

1. $0 \le P(E) \le 1$
2. $P(S) = 1$
3. For any mutually exclusive events $E_1, E_2, \ldots E_n$ (n is finite or ∞),

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i).$$

if $n=\infty$, then this series is required to converge.

Ex Flip a coin twice. $S = \{HH, HT, TH, TT\}$. Prob{getting heads once} = ?

If the coin is fair, each outcome is equally likely $\overset{\text{(Axiom 2)}}{\Rightarrow}$

$$P(\{HH\}) = P(\{HT\}) = P(\{TH\}) = P(\{TT\}) = \frac{1}{4}.$$

$E :=$ "getting heads exactly once" $= \{HT, TH\} = \{HT\} \cup \{TH\}$

$$P(E) = P(\{HT\}) + P(\{TH\}) \quad \text{(by Axiom 3)}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Ex Flip a coin until both H and T have appeared.
Prob(at least n flips) = ? $(n \ge 2)$

$$S = \left\{ \begin{array}{cccc} \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \frac{1}{2^5} \\ TH, & TTH, & TTTH, & TTTTH, \ldots \\ HT, & HHT, & HHHT, & HHHHT, \ldots \\ \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \frac{1}{2^5} \end{array} \right\}$$

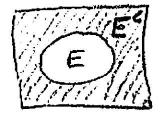
with these probabilities

$$\text{Prob(at least } n \text{ flips)} = 2\left(\frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots\right) = \frac{1}{2^{n-2}}.$$

Ex (for home): Roll two dice.
Prob{the sum of the two dice ≥ 10} $= \frac{6}{36} = \frac{1}{6}$.

2.4 Properties of probability.

Prop. 4.1 $\boxed{P(E^c) = 1 - P(E)}$



S

$S = \underbrace{E \cup E^c}_{\text{mutually exclusive}} \Rightarrow P(S) = P(E) + P(E^c)$ (by Axiom 3). QED

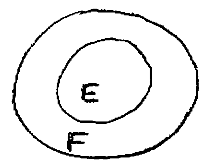
$P(S) = 1$ (by Axiom 2)

Ex (refer to Examples p.10):

$P\{\text{less than 1 flips}\} = 1 - \frac{1}{2^{n-2}}$.

$P\{\text{the sum of dice} < 10\} = 1 - \frac{1}{6} = \frac{5}{6}$.

Prop. 4.2 $\boxed{E \subset F \Rightarrow P(E) \le P(F).}$

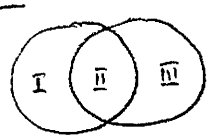


$F = \underbrace{E \cup (F \setminus E)}_{\text{mutually exclusive}} \Rightarrow P(F) = P(E) + P(F \setminus E)$ (by Axiom 3)

$\ge P(E)$ (by Axiom 2) QED

Prop. 4.3 (Inclusion-Exclusion Principle)

$$\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$



E F

$\left.\begin{array}{l} \mathrm{I} := E \setminus F, \\ \mathrm{II} := E \cap F, \\ \mathrm{III} := F \setminus E \end{array}\right\}$ mutually exclusive and $E \cup F = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$

(Show this!)

$\left.\begin{array}{l} P(E \cup F) = P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III}) \\ P(E) = P(\mathrm{I}) + P(\mathrm{II}) \\ P(F) = P(\mathrm{II}) + P(\mathrm{III}) \end{array}\right\} \Rightarrow P(E \cup F) = P(E) + P(F) - P(E \cap F).$

Q.E.D.

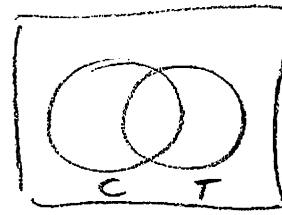
Ex. A market study produced the following results.

80% of respondents drink coffee or tea or both;

60% of respondents drink coffee;

30% of respondents drink both coffee and tea.

What percentage of respondents drink tea?



Experiment: pick a random respondent from {all respondents} = S.

Events: C = "drinks coffee", T = "drinks tea".

$$\underbrace{P(C\cup T)}_{0.8} = \underbrace{P(C)}_{0.6} + \underbrace{P(T)}_{?} - \underbrace{P(C\cap T)}_{0.3}$$

$\Rightarrow P(T) = 0.8 - 0.6 + 0.3 = 0.5.$

Ans = 50%

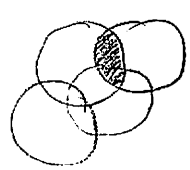
Prop 4.4 (Generalized Inclusion-Exclusion principle).

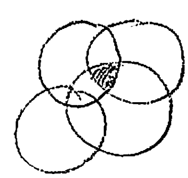
$$P(E_1 \cup \dots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}\cap E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1}\cap E_{2}\cap E_{3}) + \dots + (-1)^{n} P(E_1\cap \dots \cap E_n)$$

all intersections of pairs of events

all intersections of triples ....

the (only) intersection of all events





- QUIZ
- L. Carroll's "A Tangled Tale" (1881), Knot 10

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