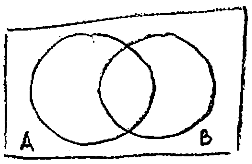
Inclusion-Exclusion Principle ctd (3.4)

Ex [L. Carroll, "A Tangled Tale" (1881), Knot 10].

(a) Simplified version:
In a furious battle, 60 percent of soldiers have lost an eye and 50 percent an ear. What percentage, at least, must have lost both?



As in Ex. p. 12,
experiment = picking a random soldier
$S = \{\text{all soldiers}\}$
A = "lost an eye", B = "lost an ear"

$P(A) = 0.6, \quad P(B) = 0.5 \qquad P(A \cap B) \geq ?$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

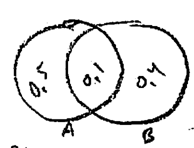
$$\Rightarrow \quad P(A \cap B) = P(A) + P(B) - \underbrace{P(A \cup B)}_{\leq 1}$$

$$\boxed{P(A \cap B) \geq P(A) + P(B) - 1}$$

$P(A \cap B) \geq 0.6 + 0.5 - 1 = 0.1$

Ans: 10%

Remark: The inequality can be attained:



Ex (b) (Full version): 70% lost an eye, 75% an ear, 80% an arm, 85% a leg. What percentage, at least, must have lost all four?

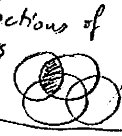
Prove $P(A \cap B \cap C \cap D) \geq P(A) + P(B) + P(C) + P(D) - 3$, apply. (next page)

~~General Inclusion-Exclusion Principle~~

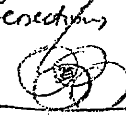
$\cancel{P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BA) - P(AC) + P(ABC)}$

~~Thm~~ $\cancel{P(E_1 \cup \dots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 < i_2 < i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) + \dots + (-1)^n \mathbb{P}(E_1 \cap \dots \cap E_n)}$

~~intersections of pairs~~



~~intersections of triples~~



Lewis Carroll. A Tangled Tale, Knot 10:

PROBLEM. In a furious battle, 70 percent of soldiers have lost an eye, 75 percent an ear, 80 percent an arm, and 85 percent a leg. Now, my dear, what percentage, at least, must have lost all four?

ANSWER. 10%

CARROLL'S SOLUTION. Adding the wounds together, we get $70 + 75 + 80 + 85 = 310$, among 100 men; which gives 3 to each, and 4 to 10 men. Therefore the least percentage is 10.

DOCTOR PETERSON'S (MATH FORUM) SOLUTION:

By inclusion-exclusion principle, $P(AB) >= P(A) + P(B) - 1$. By induction this generalizes to $P(ABCD) >= P(A) + P(B) + P(C) + P(D) - 3$. Substituting $P(A) = 0.7$, $P(B) = 0.75$, $P(C) = 0.8$, $P(D) = 0.85$, we obtain $P(ABCD) >= 0.1$.

The equality can be achieved.
To try to make that happen, we can start distributing wounds to 100 people (let's say we just give them tags identifying a wound, rather than actually doing it!), trying to keep from giving any all four until we have to. There are 310 wounds to distribute. We can use up 300 of them without giving anyone 4 wounds by giving 3 to each man (maybe give out the 70 eyes, then 30 of the 75 ears so each man has one; then start back at the first man giving out the other 45 ears and then the first 55 arms so each has 2; then start again giving out the remaining 25 arms and the first 85 legs). It really doesn't matter how we do it; but seeing that it can be done without giving anyone two of the same wound may make it clearer. We've given 300 wounds to 100 men so that none has more (or less) than 3.
Now we have 10 more wounds, which MUST be given to men who already have 3 wounds; so that is the least that can have four wounds. So the percentage (probability) is 10 of 100, or 10%.

2.5. Sample space with equally likely outcomes.

Assumptions:

Finite sample space, $S = \{1, 2, \dots, N\}$.

The outcomes are equally likely: $P(\{1\}) = P(\{2\}) = \dots\ P(\{N\}) = \frac{1}{N}$.

Then probability of <u>any</u> event E can be computed in a simple way:

$$P(E) = \sum_{i \in E} \underbrace{P(\{i\})}_{= \frac{1}{N}} = \frac{|E|}{N}.$$

$$\boxed{P(E) = \frac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in } S} = \frac{|E|}{|S|}}$$

We already used this identity: in birthday problem (p. 2), lottery problem (p. 5), letter rearrangement problem (p. 6)

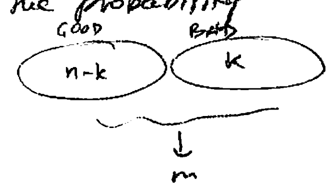
Ex Flip 2 dice $\quad P\{\text{sum} \geq 10\} = ?$

$S = \{(i,j):\ i = 1 \dots 6,\ j = 1 \dots 6\} \quad \Rightarrow \quad |S| = 36$ outcomes.

$E = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\} \Rightarrow \quad |E| = 6$ outcomes.

$\text{Ans} = \frac{6}{36} = \frac{1}{6}$.

Ex (Quality control) A shipment has n items.
m random items are tested for quality;
if at least one is defective the shipment is rejected.
If the shipment has k defective parts, what is the probability that it is rejected?



$S = \{$all possible samples of m parts chosen from n parts$\}$; $|S| = \binom{n}{m}$.

$E =$ "at least one part among m is defective" $= \{$all samples with at least one defective part$\}$.

$E^c =$ "no parts are defective" $= \{$all samples consisting of only good parts$\}$.

$$|E| = \binom{n-k}{m}.$$

$$P(E^c) = \frac{|E^c|}{|S|} = \binom{n-k}{m} \Big/ \binom{n}{m} \quad \Rightarrow \quad \text{Ans} = P(E) = 1 - P(E^c) = \boxed{1 - \binom{n-k}{m}\binom{n}{m}}$$

For example, for $n = 100$, $k = 10$:

$m = 10 \Rightarrow P(E) \simeq 0.67$
$m = 20 \Rightarrow P(E) \simeq 0.90$
$m = 25 \Rightarrow P(E) \simeq 0.95$

Ex (Elevator) 5 people enter the elevator of an 8-story building.
Each of them can exit at any floor (starting from 2) equally likely
What is the probability that they all exit at different floors?

Experiment = record the floors where each person (1, 2, 3, 4, 5) exits.

$S = \{ (f_1, f_2, f_3, f_4, f_5) : \underset{\substack{\uparrow \\ \text{floor where person } i \text{ exits.}}}{f_i} \in \{2,3,4,5,6,7,8\} \}$ (7 possibilities for each person)

$|S| = 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^5$.

E = "all exit at different floors"

$|E|$ = # ways they can exit at different floors

$= \underset{\substack{\uparrow \\ \text{person 1 exits at one of 7 floors}}}{7} \cdot \underset{\substack{\nwarrow \\ \text{person 2 exits at one of 6 floors}}}{6} \cdot 5 \cdot 4 \cdot 3 = \dfrac{7!}{2!}$, etc

$\Rightarrow P(E) = \dfrac{7!/2!}{7^5} \approx 0.15$. = Ans.

Remark: In this solution, we chose to work with an ordered group of 5 people.

Alternative solution:

Experiment' = record the 5 (unordered) numbers where the passengers exit.

$|S'| = 7^5/5!$ (disregarding the order = dividing by 5!)

$|E| = \binom{7}{5}$ (choosing 5 different floors out of 7 total)

$\Rightarrow P(E) = \dfrac{\binom{7}{5}}{7^5/5!} = \dfrac{7!/2!}{7^5}$ (same answer)