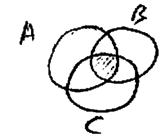
QUIZ

Lec. 6 01/18

Generalized Inclusion - Exclusion Principle

For three events:

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap A) + P(A\cap B\cap C)$



For many events:

Prop. 4.4

$$P(E_1 \cup E_2 \cup \dots \cup E_N) = \underbrace{\sum_{i=1}^{N} P(E_i)}_{\binom{N}{1}=N \text{ terms}} - \underbrace{\sum_{\substack{\text{all pairs}\\(i,j)}} P(E_i \cap E_j)}_{\binom{N}{2} \text{ terms}} + \underbrace{\sum_{\substack{\text{all triples}\\(i,j,k)}} P(E_i\cap E_j \cap E_k)}_{\binom{N}{3} \text{ terms}} - \dots + \underbrace{(-1)^{N+1} P(E_1 \cap \dots \cap E_N)}_{\binom{N}{N}=1 \text{ term}}$$

Ex. 5m (The Matching Problem)

N homeworks are returned to N students at random.
What is the probability that $\underbrace{\text{at least one student gets his/her own homework}}_{=E}$?

Represent $E = \bigcup_{i=1}^{N} E_i$, where E_i = "student i gets his/her own hw".

Use general inclusion-excl. principle above to compute $P(E)$

1) $P(E_i) = P\{\text{student } i \text{ gets his/her own HW}\}$.

$|E_i|$ = # ways the other N-1 students can get their HW $= (N-1)!$

$|S|$ = # possible ways to return N HW to N students $= N!$

$\Rightarrow P(E_i) = \dfrac{(N-1)!}{N!} = \dfrac{1}{N}$

2) $P(E_i \cap E_j) = P\{\text{students } i,j \text{ both get their own HW}\}$.

$|E_i\cap E_j|$ = # ways the other N-2 students can get their HW $= (N-2)!$

$P(E_i\cap E_j) = \dfrac{(N-2)!}{N!}$

3) $P(E_i\cap E_j \cap E_k) = \dfrac{(N-3)!}{N!}$

$$\Rightarrow P(E) = N\cdot\frac{1}{N} - \underbrace{\binom{N}{2}}_{=\frac{N!}{2!(N-2)!}}\frac{(N-2)!}{N!} + \underbrace{\binom{N}{3}}_{=\frac{N!}{3!(N-3)!}}\frac{(N-3)!}{N!} - \dots + (-1)^{N+1}\binom{N}{N}\frac{(N-N)!}{N!}$$

$$= \boxed{1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{N+1}\frac{1}{N!}}.$$

$\left(\text{Recall } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots\right)$

$\approx \underbrace{\boxed{1 - 1/e}}_{\approx 0.63}$ for large N.

Ex (GMAT club. com Problem #57447)

A fair coin is tossed 10 times. What is the probability that two heads appear consequtively?

← call this event E

Complement: E^c = "two heads do not appear consequtively".

$S = \{$all possible arrangements of 10 T/H$\}$. $|S| = 2^{10}$.

$|E^c| = ?$

Count this by fixing the total number of heads: 0, 1, 2, 3, 4, or 5.

0 head, 10 tails: T T T T T T T T T T : 1 way

1 head, 9 tails _T_T_T_T_T_T_T_T_T_ : $\binom{10}{1}$ ways

"_" denote 10 possible locations of H — choose 1 location for H

2 heads, 8 tails: _T_T_T_T_T_T_T_T_ $\binom{9}{2}$ ways

3 heads, 7 tails: _T_T_T_T_T_T_T_ : $\binom{8}{3}$ ways

4 heads, 6 tails: _T_T_T_T_T_T_ $\binom{7}{4}$ ways.

5 heads, 5 tails: _T_T_T_T_T_ $\binom{6}{5}$ ways.

$$|E^c| = 1 + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5} = 144$$

$$P(E) = 1 - P(E^c) = 1 - \frac{144}{2^{10}} \approx 0.86.$$