3.1–3.2. Conditional probabilities

- "Smoking is the leading cause of lung cancer" is a probabilistic statement.

  "Smokers are more likely to get lung cancer than non-smokers"

  is a statement about conditional probabilities:

$$P(\text{Cancer} \mid \text{Smoker}) > P(\text{Cancer} \mid \text{Non-smoker})$$

Example. Community of 352 people.

	Cancer	No
Smoker	8	32
No	16	304

$$P(\text{Cancer} \mid \text{Smoke}) = \frac{8}{8+32} = 0.2$$

$\vee$ (4 times higher)

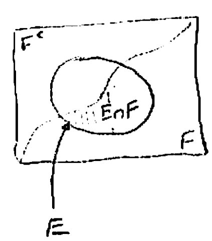
$$P(\text{Cancer} \mid \text{No smoke}) = \frac{16}{16+304} = 0.05$$

Note the calculation above. $P(\text{Cancer} \mid \text{Smoke}) = \dfrac{\overset{8}{\overset{\shortparallel}{P(\text{Cancer} \cap \text{Smoke})}}}{\underset{8+32}{\underset{\shortparallel}{P(\text{Cancer})}}}$.

Def (Conditional Probability) Consider events E, F. If $P(F) > 0$, we define

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

- F becomes our "new probability space"



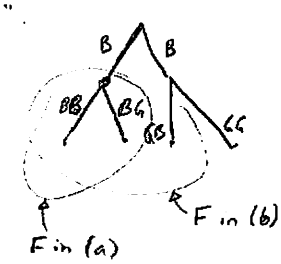
Ex Pick a family with 2 children at random (Assume all gender combinations are equally likely).

(a) $P\{\text{at least one child is a boy}\} = ?$

(b) $P\{\text{the other child is a boy} \mid \text{at least one is a boy}\} = ?$

(c) $P\{\text{the other child is a boy} \mid \text{the older is a boy}\} = ?$

"Probability tree".



$S = \{BB, BG, GB, GG\}$ (older first)

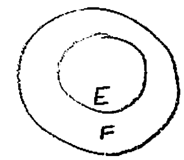
(a) Ans $= \dfrac{|\{BB, BG, GB\}|}{|S|} = \dfrac{3}{4}$.

(b) $E = \{BB\}$ $F = \{BB, BG, GB\}$. $E \cap F = \{BB\}$

Ans $= \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$ #

(c) $E = \{BB\}$, $F = \{BB, BG\}$, $E \cap F = \{BB\}$. Ans $= \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/4}{2/4} = \dfrac{1}{2}$.

Ex 75% of people live at least 70 yrs, (F)
63% of people live at least 80 yrs. (E)
What is the probability that a 70-y.old person lives at least 10 more yrs?



$P(E) = 0.63, \quad P(F) = 0.75,$

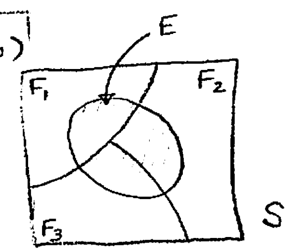
$$P(E|F) = \frac{P(E\cap F)}{P(F)} \underset{E\subset F}{=\!=} \frac{P(E)}{P(F)} = \frac{0.63}{0.75} = 0.84.$$

The Law of Total Probability

Allows to compute $P(E)$ from $P(E|F)$ ← additional information

Note by def.: (∗) $\boxed{P(E\cap F) = P(E|F)\,P(F)}$ ("Multiplication Rule")

Prop (Law of Total Prob.) Assume that $F_1, F_2, \dots, F_N$ ($N<\infty$ or $N=\infty$) are mutually exclusive events such that
$$S = \bigcup_{i=1}^{N} F_i .$$
Then for every event E,
$$P(E) = \sum_{i=1}^{N} P(E|F_i)\,P(F_i)$$



Proof By assumptions, $E\cap F_i$ are mutually disjoint and
$$E = \bigcup_{i=1}^{N} (E\cap F_i).$$
Hence (by Axiom 3 of Probability),
$$P(E) = \sum_{i=1}^{N} P(E\cap F_i) = \sum_{i=1}^{N} P(E|F_i)P(F_i) \quad \text{(by (∗) above)} \qquad \text{QED}$$

Ex 70% of Delta flights depart on time.
80% of flights that depart on time arrive on time.
90% of flights that depart late arrive late.
What % of flights arrive on time?

Choose a random flight. Events: D = "departs on time", A = "arrives on time"

$P(A) = P(A|D)\,P(D) + P(A|D^c)\,P(D^c)$ (by L.T.P.)
$= 0.8\cdot 0.7 + (1-0.9)(1-0.7) = 0.59$ ANS = 59%.

Ex The probability that twins have same gender ≈ 0.64

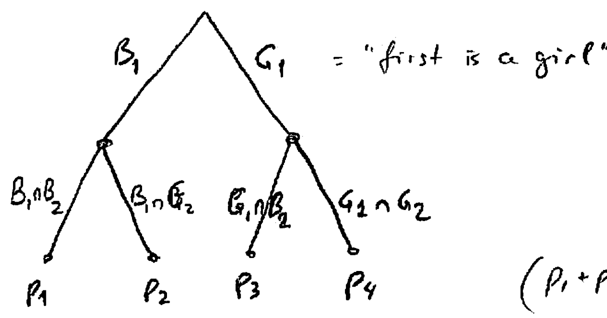
The probability that a newborn is a boy ≈ 0.51.

What is the probability that the twins are boys if the first of them is a boy?

B_1 = "first of the twins is a boy", B_2 = "second is a boy".

$$P(B_2 | B_1) = \frac{P(B_1 \cap B_2)}{P(B_1)}. \qquad P(B_1) = 0.51$$

$P(B_1 \cap B_2) = ?$ Use the "probability tree".



$(p_1 + p_2 + p_3 + p_4 = 1)$

$p_1 = ?$ We know:

$$\begin{cases} p_1 + p_4 = P(\text{twins have same gender}) = 0.64 \\ p_1 + p_2 = P(B_1) = 0.51 \\ p_2 + p_4 = P(G_2) = 1 - 0.51 = 0.49. \end{cases}$$

$$\Rightarrow \quad p_1 = \frac{0.64 + 0.51 - 0.49}{2} = 0.33.$$

$$\Rightarrow \quad \text{Ans} = \frac{P(B_1 \cap B_2)}{P(B_1)} = \frac{0.33}{0.51} \approx 0.647$$

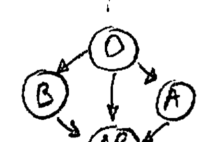
(O, A, B, AB)

Ex In blood transfusion, the blood types of the donor and the recipient must be taken into account.

Recipients with AB type can receive blood of any type;

Recipients with A or B types can receive blood of the same type and also O;

Recipients with O type can only receive the same type.



34% of population have O type, 38% A type, 20% B type, 8% AB type.

Find the probability that a $\underbrace{\text{random donor can give blood to a random recipient}}_{E}$

prob. in question can be computed by law of tot. prob. by specifying the type of recipient.

$$P(E) = P(E | \underset{\text{recipient}}{O}) P(O) + P(E | A) P(A) + P(E | B) P(B) + P(E | AB) P(AB)$$

$$= P(O) \cdot P(O) + [P(A) + P(O)] P(A) + [P(B) + P(O)] P(B) + 1 \cdot P(AB) = 0.57.$$