3.4 Independent Events.

- Intuitive def. of independence of E, F:

$$P(E|F) = P(E), \qquad P(F|E) = P(F) \qquad (*)$$

i.e. F does not affect the likelihood of E, and vice versa.

Ex: Flip coin 10 times.

$$P\{\underbrace{10\text{'th flip} = H}_{E} \mid \underbrace{\text{first 9 flips} = H}_{F}\} = \frac{1}{2}$$ because E, F independent.

- The identities in $(*)$ can be written as

$$\frac{P(E \cap F)}{P(F)} = P(E), \qquad \frac{P(F \cap E)}{P(E)} = P(F).$$

SAME!

Def Events E, F are __independent__ if $P(E \cap F) = P(E)\,P(F)$

- Independence implies $(*)$
- __Warning__: independent $\neq$ mutually exclusive

Def Events $E_1, E_2, \ldots$ are independent if

$(+)$ $P(E_i \cap E_j) = P(E_i)\,P(E_j)$ for all pairs (i,j) of distinct integers i, j;

$P(E_i \cap E_j \cap E_k) = P(E_i)\,P(E_j)\,P(E_k)$ for all triples $(i,j,k), \ldots$

etc....

- __Note__: pairwise independence (identities $(+)$ only) does not imply independence

__Example__: roll 3 dice

$E = \{\text{first die} = 3\}$, $F = \{\text{second die} = 4\}$, $G = \{\text{sum of all three} = 7\}$.

Check that E, F are indep; F, G indep, E, G indep, but E, F, G are not indep.

Ex A plane has 4 engines. It can fly if 2 or more engines are functioning.
Each engine fails independently with probability p
$P\{\text{plane flies}\} = ?$

$E_i = \{\text{engine } i \text{ functions}\}$.

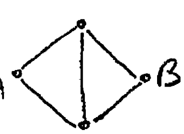
$P\{\text{plane flies}\} = 1 - P\{\text{at most one engine functions}\}$

$$= 1 - P\{\text{no engine functions}\} - P\{\text{one functions}\}.$$

$$= \boxed{1 - (1-p)^4 - \underbrace{\binom{4}{1}}_{\substack{\text{choosing the}\\ \text{functioning}\\ \text{engine}}} \underbrace{p\,(1-p)^3}_{\text{Prob}\{\text{the given engine functions others fail}\}}}$$

Computing probabilities by conditioning

Ex (Networks). Consider the network:



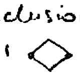
Each link fails with prob. p independently.
Find the prob. that $\underbrace{A \text{ and } B \text{ are connected}}_{=C}$.

Condition on the state of the vertical link; $V = \{\text{vertical link works}\}$.

$$P(C) = P(C|V)\underbrace{P(V)}_{=1-p} + P(C|V^c)\underbrace{P(V^c)}_{=p}$$

1) $P(C|V) = ?$ If V occurs, the network can only fail if $\overbrace{A \text{ is not connected to the vert link}}^{\text{prob}=p^2}$, or $\underbrace{B \text{ is not connected to vert. link, or both}}_{\text{prob}=p^2}$

$$P(C^c|V) = \underbrace{p^2}_{A<} + \underbrace{p^2}_{>B} - \underbrace{p^4}_{\text{all 4 links fail}}. \quad (\text{by inclusion-excl.})$$



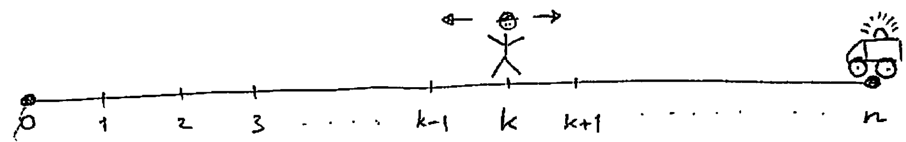
2) $P(C|V^c) = ?$ If V^c occurs, the network looks like this: A  B

Network works if either $\underbrace{\text{both top links work}}_{\text{prob}=(1-p)^2}$, or $\underbrace{\text{both bottom links work}}_{\text{prob}=(1-p)^2}$, or $\underbrace{\text{both}}_{\text{prob}=(1-p)^4}$.

$$\Rightarrow P(C|V^c) = (1-p)^2 + (1-p)^2 - (1-p)^4.$$

Hence: $P(C) = (1-2p^2+p^4)(1-p) + [2(1-p)^2 - (1-p)^4]\,p = 1-2p^2-2p^3+5p^4-2p^5$

Example (Simple random walk)



A particle is placed at k.
Each second, the particle moves 1 step to the left or to the right independently with prob $1/2$ each.
What is the probability that $\underbrace{\text{the particle reaches } n \text{ before reaching } 0}_{E_k}$?

Condition on the first step, L or R.

$$P(E_k) = P(E_k \mid L)\,P(L) + P(E_k \mid R)\,P(R)$$

$$= P(E_{k-1}) \cdot \frac{1}{2} + P(E_{k+1}) \cdot \frac{1}{2}$$

(Conditioned on L, the game "resets" with particle at $k-1$ instead of k)

Denoting $p_k = P(E_k)$, we obtain

$$\begin{cases} p_k = \frac{1}{2}(p_{k-1} + p_{k+1}), & k = 1, \dots, n-1 \\ p_0 = 0, \quad p_n = 1 \end{cases}$$

$n+1$ linear equations with $n+1$ unknowns. Solving (do this!) gets us

$$\boxed{p_k = \frac{k}{n}}$$

- Interpretations:

  (a) Finance: 0 = bankrupcy, n = payoff, k = initial capital.
  P(payoff before bankrupcy) = ?

  For this example, a biased random walk is more relevant, where $P(R) = p$, $P(L) = 1-p$ for some $0<p<1$ (See Ross Ex 4l).

  (b) Recurrence: for $n \to \infty$, $p_k \to 0$ (with k fixed).
  $\Rightarrow$ with prob. 1 ("almost surely"), the particle will visit any given site (in this case)
  (If lost, will return home by random walk).