4.1. Random variables

Conduct an experiment.

To each outcome assign a number.

Such assignment is called a r.v.

Def A r.v. X is a real-valued function on the sample space S.

Ex (a) Experiment. flip 3 coins.

$$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

$$\begin{array}{cccccccc} TTT & TTH & THT & THH & HTT & HTH & HHT & HHH \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \end{array}$$

look for # heads in each outcome

This assignment, "the number of heads" is a random variable X.

Note: individual outcomes are not important, the value of X is important (e.g. TTH, THT, HTT all give $X=1$).

Ex Other examples of random variables:

(b) Age of a randomly chosen person in A.A.

(c) UM's enrollment in Fall 2012

(d) Lifetime of a computer you are going to buy;

(e) Delay of a flight,

(f) Molecule's velocity at a given moment;

(g) An imperfect measurement of the distance between to given pts on Earth (or any other kind of measurement)

(h) Wait time in a line.

4.2 Discrete r.v.'s; pmf

Def A r.v. X is discrete if X takes on a finite or countable # of values $x_1, x_2, \ldots$

Ex : (a), (b) (round off), (c) are discrete; the others are not.

Def (PMF) Let X be a discrete r.v.
The probability mass function (pmf) of X is defined as

$$p(x) = \mathbb{P}\{X = x\}, \qquad x \in \{\underbrace{x_1, x_2}_{\text{values of } X}, \ldots\}$$

· Note : in some literature, $p(a)$ is defined for all $a \in \mathbb{R}$, but in that case $p(a) = 0$ if $a \notin \{x_1, x_2, \ldots\}$.

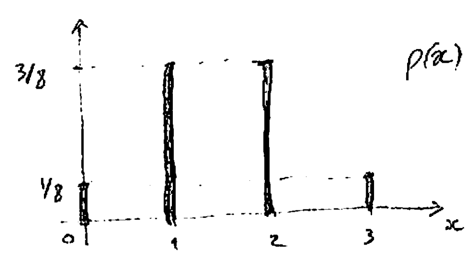
Property :

$$\boxed{\sum_i p(x_i) = 1}$$

$$\left(\text{indeed, } S = \bigcup_i \{X = x_i\} \implies \underset{\substack{\| \\ 1}}{P(S)} = \underset{\substack{\| \\ \sum p(x_i)}}{\sum P\{X = x_i\}}\right).$$

Ex X = # heads in 3 coin flips (Ex. a p. 27)

$$p(0) = \frac{1}{8}; \quad p(1) = \frac{3}{8}, \quad p(2) = \frac{3}{8}, \quad p(3) = \frac{1}{8}$$



Ex X = # heads in n coin flips

$$p(k) = \binom{n}{k} \cdot \left(\frac{1}{2}\right)^n, \qquad k = 0, 1, \ldots, n$$

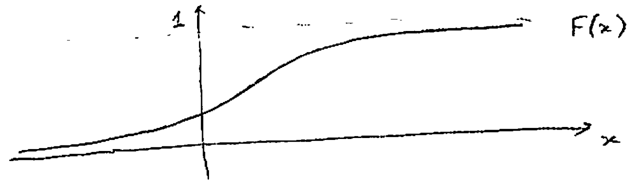
$\binom{n}{k}$: # ways to choose a pattern (e.g. TTHHHTHT...T)

$\left(\frac{1}{2}\right)^n$: probability to flip a specific pattern

CDF

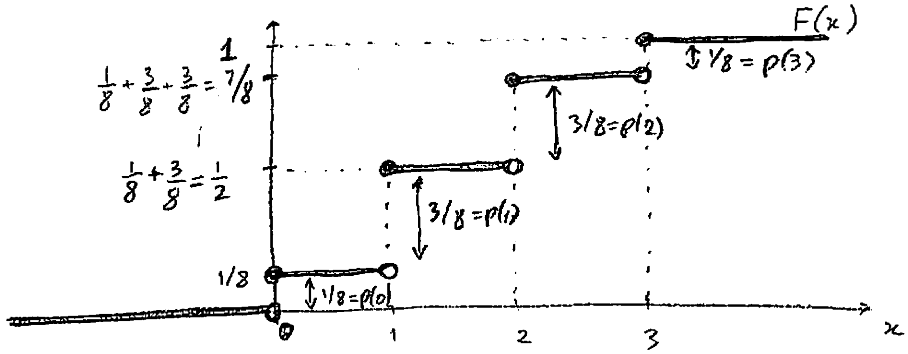
Def (CDF). Let X be a random variable.
The cumulative distribution function (cdf) of X is defined as

$$F(x) = \mathbb{P}\{X \le x\}, \qquad x \in \mathbb{R}.$$



- **Properties:**
  1) F is non-decreasing;
  2) $\lim_{x\to-\infty} F(x) = 0, \qquad \lim_{x\to\infty} F(x) = 1$

Ex $X = \#$ heads in 3 coin flips (Ex (a) p. 27)



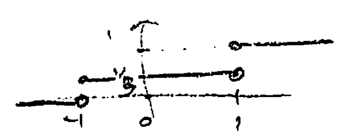
$$F(x) = \begin{cases} 0, & x < 0 \\ 1/8, & 0 \le x < 1 \\ 1/2, & 1 \le x < 2 \\ 7/8, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Connection between pmf and cdf:

pmf → cdf: $F(x) = \mathbb{P}\{X \le x\} = \sum_{x_i \le x} \mathbb{P}\{X = x_i\} = \sum_{x_i \le x} p(x_i)$

cdf → pmf: $p(x_i) \ne 0$ if x_i is a point of discontinuity of F,
the jump of F at x_i equals $p(x_i)$.

Ex Let X be a r.v. with cdf $F = \begin{cases} 0, & x < -1 \\ 1/3, & -1 \le x < 1 \\ 1, & x \ge 1 \end{cases}$ Find pmf.



$p(-1) = \frac{1}{3} - 0 = \frac{1}{3}$; $\quad p(1) = 1 - 1/3 = 2/3$.

Note: The "distribution" of X is the knowledge of "what values X takes with what probabilities".
Thus either pmf $p(x)$ or cdf $F(x)$ describe the distribution of X.