Lec. 15 02/08/2012

Binomial & Poisson distributions (continued)

Remark : Since pmf of Poisson (λ) is $p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2 \ldots$

and $\sum_{k=0}^{\infty} p(k) = 1$ (see p.28), we have

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 1$$

$$\Rightarrow \boxed{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}} \quad , \quad \text{Taylor series of } e^{\lambda}$$

Ex: A meteorite that hits a weather satellite causes its failure with prob. p.
k meteorites hit a satellite per day, on average.
What is the prob that satellite will fail during a year ?

$365k$ meteorites hit the satellite during a year, on ave.

$365k \cdot p$ meteorites will cause failure during a year, on ave.

X = # meteorites causing failure during a year.

$X \sim$ Poisson (λ), $\quad \lambda = E\{X\} = 365 k \cdot p$.

$$P\{\text{failure}\} = 1 - P\{X=0\} = 1 - e^{-\lambda} = 1 - e^{-365kp}$$

For example, if $k = 1.5$, $p = 0.001$ then $P\{\text{failure}\} \approx 0.42$

Ex

A court sends $\overset{=32}{N}$ jury duty letters to $\overset{=8}{n}$ zip codes, independently at random.

(a) What is the expected value of $\underbrace{\text{\# of letters sent to each zip}}_{= X_i}$?

$$X_i \sim \text{Binom}\left(N, \frac{1}{n}\right) \Rightarrow E[X_i] = \frac{N}{n}$$

$$\text{If } N = 32,\ n = 8 \Rightarrow E[X_i] = 4.$$

(b) What is the expected number of "underrepresented" zip codes — those where at most 2 letters are sent?

Let $Z :=$ # of zip codes where k letters are sent

Method of indicators: $Z = \sum_{i=1}^{n} Z_i$, where $Z_i = \begin{cases} 1, & \text{if } k \text{ letters are sent to zip } i \\ 0, & \text{otherwise.} \end{cases}$

$$E[Z] = \sum_{i=1}^{n} E[Z_i] = n \cdot \underbrace{E[Z_i]}_{= ?}.$$

$Z_i \sim \text{Bernoulli}(p)$,

$$p = P\{Z_i = 1\} = P\{k \text{ letters to zip } i\} = P\{X_i = k\}$$

$$= \binom{N}{k}\left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{N-k} \qquad (\text{as } X_i \sim \text{Binom}(N, 1/n))$$

$$\Rightarrow E[Z_i] = p \Rightarrow E[Z] = np = \boxed{n\binom{N}{k}\left(\frac{1}{n}\right)^k\left(1 - \frac{1}{n}\right)^{N-k}}$$

Illustration: let $N = 32$, $n = 8$

k	$E[Z]$
0	0.11
1	0.51
2	1.13

(see a complete histogram on the next page).

Hence: $E(\text{\# zips with} \le 2 \text{ letters}) \approx 0.11 + 0.51 + 1.13 \approx 1.75$.

Mathematica 8 Calculation (for illustration of Ex. p 45)

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(* N letters are sent to n zip codes independently at
 random. Find the expected number of zip codes that received k letters. *)

In[55]:= EZ[N_, n_, k_] := n * Binomial[N, k] * (1 / n) ^k * (1 - 1 / n) ^ (N - k);

(* Plot the answer for N=32, n=8, k=0,1,...,10. *)

In[58]:= f[k_] := EZ[32, 8, k];

In[57]:= DiscretePlot[f[k], {k, 0, 10}, ExtentSize → Full]
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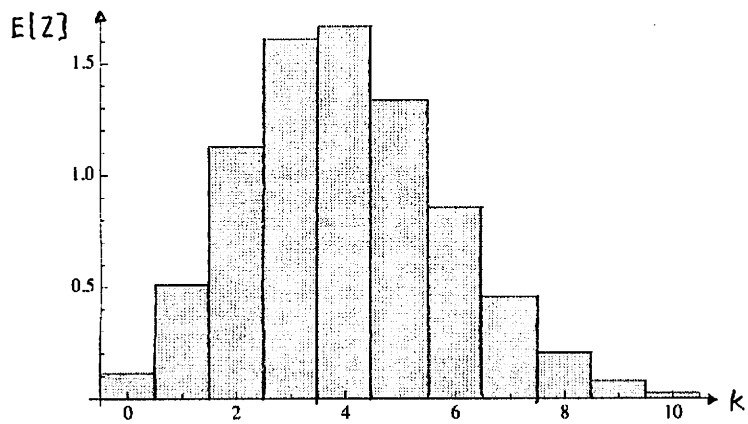


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(* The expected number of zip codes that received at most 2 letters each *)

In[50]:=
 N[f[0] + f[1] + f[2]]

Out[50]= f[32., 8., 0.] + f[32., 8., 1.] + f[32., 8., 2.]

(* Compute the standard deviation in the original problem *)

In[24]:= EZsquare[N_, n_, k_] := n * Binomial[N, k] * (1 / n) ^k * (1 - 1 / n) ^ (N - k) +
    (n ^ 2 - n) * Binomial[N, k] * Binomial[N - k, k] * (1 / n) ^ (2 k) * (1 - 2 / n) ^ (N - 2 k);

In[33]:= VarZ[N_, n_, k_] := EZsquare[N, n, k] - (EZ[N, n, k]) ^ 2;

In[34]:= SdevZ[N_, n_, k_] := (VarZ[N, n, k]) ^ (1 / 2);

(* The standard deviation of the number of zip codes that received no letters *)

In[74]:= N[SdevZ[32, 8, 0]]

Out[74]= {0.323586}
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