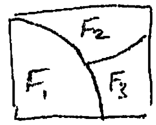
Computing expectation by conditioning. (~7.5).

- Recall computing probabilities by conditioning:

Suppose $S = \bigcup_i F_i$ is a decomposition into mutually exclusive events F_i



Then Law of Total Probability (p.20) states:

for every event E,

$$P(E) = \sum_i P(E \cap F_i) = \sum_i P(E \mid F_i)\, P(F_i).$$

- There is a similar:

Thm (Law of Total Expectation):

$$\boxed{E[X] = \sum_i E[X \mid F_i] \cdot P(F_i)}$$

Here $E[X \mid F]$ refers to the conditional expectation of X given event F, defined as (for discrete X)

$$\boxed{E[X \mid F] = \sum_x x\, P\{X = x \mid F\}}$$

(recall that $E[X] = \sum_x x\, P\{X = x\}$)

Proof of law of Total Exp.

$$E[X] = \sum_x x\, P\{X = x\} \quad \text{(by def)}$$

$$= \sum_x x \sum_i P\{X = x \mid F_i\} P(F_i) \quad \text{(by law of Total Prob.)}$$

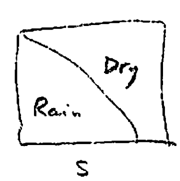
$$= \sum_i \left(\sum_x x\, P\{X = x \mid F_i\} \right) P(F_i) = \sum_i E[X \mid F_i] \cdot P(F_i).$$

QED.

Ex The average # of traffic accidents in Berkeley on a rainy day is 9,
on a dry day is 3.

It rains with prob. 0.2 in Berkeley.

What is the ave # of accidents per day in Berkeley? (X)

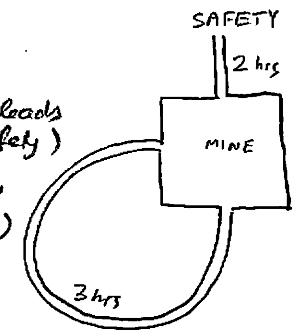


$$E[X] = E[X \mid \text{Rain}]\, P(\text{Rain}) + E[X \mid \text{Dry}] \cdot P(\text{Dry}) \quad \text{(by law of Total Exp)}$$

$$= 9 \cdot 0.2 + 3 \cdot 0.8 = 4.2.$$

(Ch.7, Ex.5c)

Ex A miner is trapped in a mine with three doors.
One door leads to a tunnel which takes the miner to safety in 2 hrs of travel.
The other two doors are connected by a loop, which one can pass in 3 hrs.
The miner is equally likely to choose any of the two doors at any time. ("memoryless").
What is the expected $\underbrace{\text{time it takes to reach safety}}_{= X}$?



Condition on the miner's initial decision — "G": open the good door (that leads to safety); "B": open a bad door (connected by the loop)

$$E[X] = \underbrace{E[X \mid G]}_{=2}\ \underbrace{P(G)}_{=1/3} + \underbrace{E[X \mid B]}_{=3+E[X]}\ \underbrace{P(B)}_{=2/3}$$

(after 3 hrs, the miner starts over again)

$$E[X] = \frac{2}{3} + \frac{2}{3}(3+E[X])$$

Solving yields

$$E[X] = 8 \text{ hrs}$$

4.8.1. Geometric distribution

"Time until first success":

Def Consider an (infinite) sequence of indep. trials, with p = prob. of success in each trial.
Let X = # of the trial that is the first success.
Then X is called a geometric r.v.; notation
$$X \sim \text{Geom}(p).$$

- PMF:
$$p(k) = P\{X=k\} = \underbrace{(1-p)^{k-1}}_{\text{prob. that the first } k-1 \text{ trials are failures}} \cdot \underbrace{p}_{\text{prob that the } k\text{'th trial is a success}}, \quad k = 1, 2, \ldots$$

Remark: Since $\sum_{k=1}^{\infty} p(k) = 1$,

$$\sum_{k=1}^{\infty} (1-p)^{k-1} p = 1 \quad \Rightarrow \quad \boxed{\sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{p}} \qquad \text{(Geometric series).}$$

- $E[X] = ?$ Condition on the outcome of the first trial; S or F.

$$E[X] = \underbrace{E[X|S]}_{=1} \underbrace{P(S)}_{=p} + \underbrace{E[X|F]}_{=1+E[X]} \cdot \underbrace{P(F)}_{=1-p}$$

$1 + E[X]$ (after 1 failure, we start over again). ("memoryless property")

$$E[X] = p + (1+E[X])(1-p).$$

$$\Rightarrow \boxed{E[X] = \frac{1}{p}}$$

Ex (referring to Ex. p. 48). $\underbrace{\text{How many doors on average will the miner open,}}_{Y}$ until safety?

$Y \sim Geom(\frac{1}{3}) \Rightarrow E[Y] = 3$ doors.

(Quality control).

Ex Each time a notebook is manufactured, a fault may occur in the assembly line with prob. $p = 0.05$, which will cause defects in the laptop being manufactured and all further laptops.
The quality control department checks all manufactured laptops; a defect is found in a laptop with prob. $q = 0.9$
When a defect is found, the laptop is disposed of, and the line gets repaired.
What is % of defective laptops produced?

$$\underbrace{G\,G\,G\,G\,G\,G\,G\,B}_{X}\!\!\!\!\overbrace{}^{}\underbrace{B\,B\,\cancel{B}}_{Y}\ G\,G\,G\,G\ \ldots\ldots$$

fault occurs, defect found,

G = "Good", B = "Bad"

X = # of laptops until fault occurs;
Y = # of bad laptops until defect found.

$X \sim Geom(p)$, $Y = Geom(q)$. $E[X] = 1/p$, $E[Y] = 1/q$.

Between two consecutive line repairs:

# of good laptops $= X - 1$, # of bad laptops $= Y - 1$ (1 is disposed of).

$E[\cdot] = \frac{1}{p} - 1$, $E[\cdot] = \frac{1}{q} - 1$.

Ans: $\dfrac{\frac{1}{q} - 1}{\frac{1}{p} - 1 + \frac{1}{q} - 1} \approx 0.058$ $\approx 0.6\%$

Ex (for home): $X \sim Geom(p) \Rightarrow \boxed{Var(X) = \frac{1-p}{p^2}}$ (Textbook Ex. 7.5h).