5.1 Continuous distributions

Examples : · lifetime of equipment
· travel time between two points (in city or on highway)
· payoff of an enterprise
...

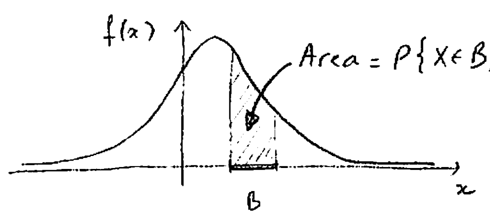
· pmf does not make sense : $p(x) = P\{X = x\} = 0$ for all x.

→ pdf :

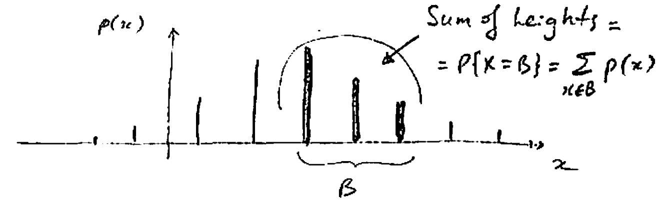
Def A r.v X is continuous if $\exists\, f(x) \geq 0$ defined on $\mathbb{R}$ s.t.

$$P\{X \in B\} = \int_B f(x)\,dx \qquad \text{for each set } B \subseteq \mathbb{R}$$

f is called pdf ("probability density function", or just "density") of X.



Compare to discrete distributions:



Remarks : 1) For $B = [a, b]$, we have

$$P\{a \leq X \leq b\} = \int_a^b f(x)\,dx.$$

2) For any $a \in \mathbb{R}$, we have

$$P\{X = a\} = \int_a^a f(x)\,dx = 0.$$

3) $$\int_{-\infty}^{\infty} f(x)\,dx = P\{-\infty < X < \infty\} = 1.$$

Connection with cdf :

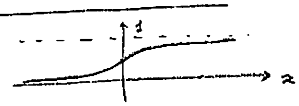
1) $$F(a) = P\{X \leq a\} = \int_{-\infty}^{a} f(x)\,dx$$

Differentiating (by Fundamental Thm of Calculus) ⇒

$$F'(a) = f(a)$$

2) $$P\{a \leq X \leq b\} = P\{X \leq b\} - P\{X \leq a\} = F(b) - F(a)$$

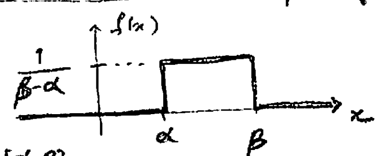
SUMMARY:

pdf $f(x)$	cdf $F(x)$
	
$f \geq 0$, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.	$F(-\infty)=0$, $F(\infty)=1$, F non-decreasing
$P\{a \leq X \leq b\} = \int_a^b f(x)\,dx$	$P\{a \leq X \leq b\} = F(b) - F(a)$.
$f(a) = F'(a)$	$F(a) = \int_{-\infty}^{a} f(x)\,dx$

~ 5.3. Uniform distribution.

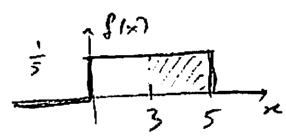
Def We say that a r.v X has the uniform distribution on $[\alpha, \beta]$ if X has pdf

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & x \in [\alpha,\beta] \\ 0, & \text{elsewhere} \end{cases}$$



- Meaning: X is equally likely to take values anywhere in $[\alpha, \beta]$.
- Why $\frac{1}{\beta-\alpha}$? Because one must have $1 = \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\beta-\alpha}\int_\alpha^\beta dx = 1$.

Ex Buses arrive at a bus stop at 5-minute intervals (on average). When you come to the bus stop what is the prob. that you will have to wait more than 3 minutes?

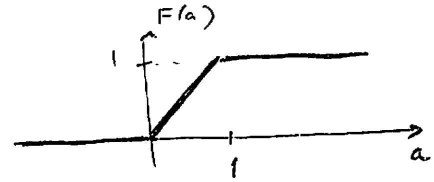


X = your wait time $\sim \text{Unif}[0,5]$.

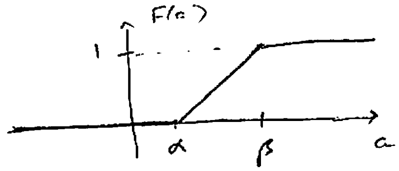
$$P\{X > 3\} = \int_3^\infty f(x)\,dx = \int_3^5 \frac{dx}{5} = \frac{2}{5} = 0.4$$

cdf: • $X \sim \text{Unif}[0,1]$ $\Rightarrow$

$$F(a) = \int_{-\infty}^{a} f(x)\,dx = \begin{cases} 0, & a < 0 \\ a, & a \in (0,1) \\ 1, & a > 1 \end{cases}$$

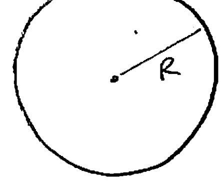


• $X \sim \text{Unif}[\alpha, \beta) \Rightarrow$



Ex(a) (2D uniform distribution).

A certain city has the shape of a circle with radius $R = 10$ mi and it is uniformly populated.
Choose a random person from the city.
What is the prob. that s/he lives within 5 miles from the center?



(Not 1/2)!

X = distance to center from a randomly chosen person (random point in the circle)

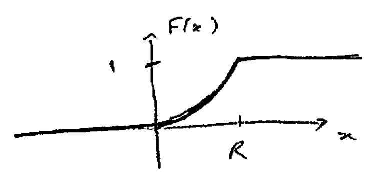
$X \not\sim \text{Unif}[0, R]$!

% of population that lives in a certain area B of the city is proportional to the area of B.

$$P\{X \leq 5\} = \frac{\text{Area (circle of radius 5)}}{\text{Area (circle of radius 10)}} = \frac{\pi \cdot 5^2}{\pi \cdot 10^2} = \frac{1}{4}.$$

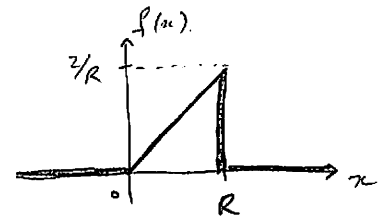
(b) Compute cdf of X (the dist. to center)

$$F(x) = P\{X \leq x\} = \frac{\pi x^2}{\pi R^2} = \begin{cases} \frac{x^2}{R^2}, & 0 \leq x \leq R \\ 0, & x < 0 \\ 1, & x > R \end{cases}$$



(c) Compute pdf of X

$$f(x) = F'(x) = \begin{cases} \frac{2x}{R^2}, & 0 \leq x \leq R \\ 0, & \text{elsewhere} \end{cases}$$



⇒ Distribution of X is not uniform, skewed toward the city perimeter.

Remark (meaning of pdf):

$f(a)$ is proportional to the probability that X is near a:

$$P\left\{a - \frac{\varepsilon}{2} \leq X \leq a + \frac{\varepsilon}{2}\right\} = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(x)\,dx \approx \varepsilon \cdot f(a)$$