Transformations of random variables

($\sim$ 6.7 but in one variable)

- If we know the distr. of X, how to compute the distr. of $Y = g(X)$?
  ($\uparrow$ a function of r.v. X)

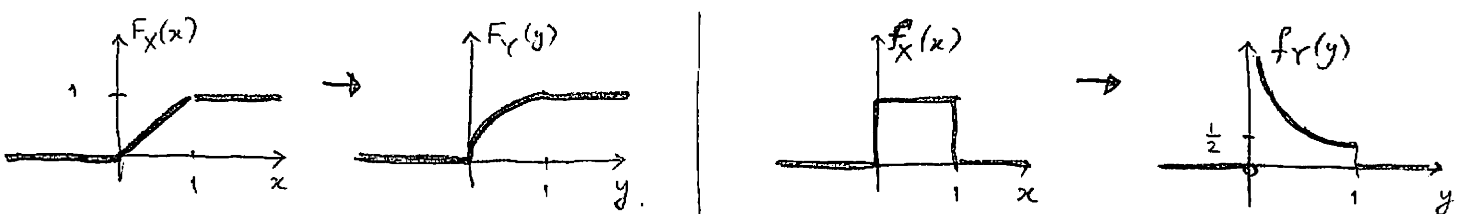
Ex $X \sim \text{Unif}[0,1]$. Compute the cdf and pdf of $Y = X^2$.

cdf : $F_Y(y) = P\{Y \le y\} = P\{X^2 \le y\}$ $\quad (0 \le y \le 1)$

$= P\{X \le \sqrt{y}\} = \sqrt{y}$

Thus $F_Y(y) = \begin{cases} 0, & y < 0 \\ \sqrt{y}, & 0 \le y \le 1 \\ 1, & y > 1. \end{cases}$

pdf : $f_Y(y) = \dfrac{d}{dy} F_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$



CDF $\qquad\qquad$ PDF

- Example of application: a picture with dimensions $X \times X$ pixels is stored as a file of size $\sim Y = X^2$ Mb.
  $f_Y(y)$ is the distr. of file sizes on a computer (when dim. $X \sim$ Unif).

- More generally:

cdf : $F_Y(y) = P\{g(X) \le y\} = P\{X \le g^{-1}(y)\}$ $\quad$ (when g is monotone increasing)

$= F_X(g^{-1}(y))$

pdf : $f_Y(y) = \dfrac{d}{dy} F_Y(y) = \dfrac{d}{dy} F_X(\underbrace{g^{-1}(y)}_{=x}) = \dfrac{d}{dx} F_X(x) \cdot \dfrac{d}{dy} g^{-1}(y)$ $\quad$ (by Chain rule)

$= f_X(x) \cdot \left(\dfrac{d}{dx} g(x)\right)^{-1}$ $\quad$ (by the derivative of inverse function).

- For g decreasing, same but with "$-$" sign.

Thm Let X be a r.v. with pdf $f_X(x)$, let $g(x)$ be an increasing function on some interval I, let $Y = g(X)$. Then pdf of Y is

$$\boxed{f_Y(y) = f_X(x) \cdot \left|\frac{d}{dx} g(x)\right|^{-1} \quad \text{where } x = g^{-1}(y) \in I}$$

abs. value.

Ex $X \sim \text{Unif}[0,1]$, $Y = X^2$ — let's use Thm with $g(x) = x^2$ on $[0,1]$

$$f_Y(y) = \underbrace{f_X(x)}_{=1} \cdot |2x|^{-1} = \frac{1}{2x} \underset{y=x^2}{=} \frac{1}{2\sqrt{y}}, \quad y \in [0,1]$$

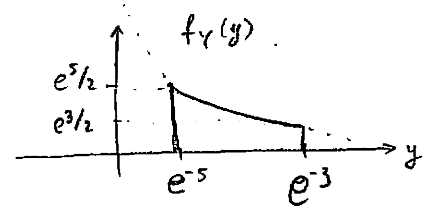
(Same answer as on p.53 but faster).

Ex Concentration of a drug in blood after t hours is e^{-t}.
The concentration is measured at a random time $\sim \text{Unif}[3,5]$
Find pdf of concentration

$X \sim \text{Unif}[2,3]$, $Y \sim e^{-X}$ monotone on $[3,5]$

$$f_Y(y) = \underbrace{f_X(x)}_{=\frac{1}{2}} \cdot |-e^{-x}|^{-1} = \frac{1}{2} e^{x} \underset{y=e^{-x}}{=} \frac{1}{2y} \quad \text{where } x \in [3,5] \Leftrightarrow y \in [e^{-5}, e^{-3}]$$

Thus $f_Y(y) = \begin{cases} \frac{1}{2y}, & y \in [e^{-3}, e^{-5}] \\ 0, & \text{elsewhere} \end{cases}$



__Application__: Simulating random variables with a given distribution, from $X \sim \text{Unif}[0,1]$.

Lec. 19, 02/17

5.2. Expectation, variance of continuous r.v's.

Def For a r.v. X with pdf $f(x)$, the expected value is defined as
$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

- Compare to $E[X] = \sum_x x\, p(x)$ for discrete r.v. X.

Ex $X \sim \text{Unif}[0,1]$. $E[X] = \int_0^1 x \cdot 1\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

Ex (Refer to Ex. p.52). Compute the average dist. to center from a point in the circle of radius R

X = dist to center. Recall $f(x) = \begin{cases} \frac{2x}{R^2}, & 0 \le x \le R \\ 0, & \text{elsewhere} \end{cases}$

$$E[X] = \int_0^R x \cdot \frac{2x}{R^2}\, dx = \frac{2}{R^2}\int_0^R x^2\, dx = \frac{2}{R^2}\frac{R^3}{3} = \frac{2}{3}R$$

$> \frac{1}{2}R$!