**Ex** $X \sim \text{Unif}[0,1]$, $Y = X^2$ — let's use Thm with $g(x) = x^2$ on $[0,1]$

$$f_Y(y) = \underbrace{f_X(x)}_{=1} \cdot |2x|^{-1} = \frac{1}{2x} \underset{y=x^2}{=} \frac{1}{2\sqrt{y}}, \quad y \in [0,1]$$

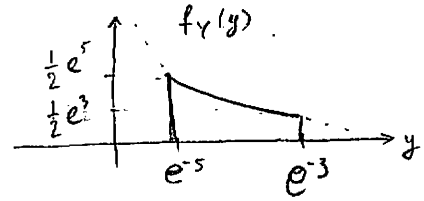
(Same answer as on p.53 but faster).

**Ex** Concentration of alcohol in blood after $t$ hours is $e^{-t}$.
The concentration is measured at a random time $\sim \text{Unif}[3,5]$
Find pdf of concentration

$X \sim \text{Unif}[2,3]$, $Y \sim e^{-X}$ monotone on $[3,5]$

$$f_Y(y) = \underbrace{f_X(x)}_{=\frac{1}{2}} \cdot |-e^{-x}|^{-1} = \frac{1}{2} e^{x} \underset{y=e^{-x}}{=} \frac{1}{2y} \quad \text{where } x \in [3,5] \iff y \in [e^{-5}, e^{-3}]$$

Thus $f_Y(y) = \begin{cases} \frac{1}{2y}, & y \in [e^{-3}, e^{-5}] \\ 0, & \text{elsewhere} \end{cases}$




__Application__: Simulating random variables with a given distribution, from $X \sim \text{Unif}[0,1]$.

Lec. 19, 02/17

## 5.2. Expectation, variance of continuous r.v's.

**Def** For a r.v $X$ with pdf $f(x)$, the expected value is defined as

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

- Compare to $E[X] = \sum_x x\, p(x)$ for discrete r.v. $X$.

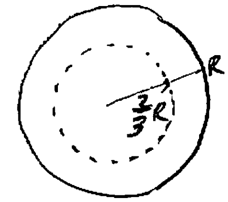
**Ex** $X \sim \text{Unif}[0,1]$. $E[X] = \int_0^1 x \cdot 1\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

**Ex** (Refer to Ex. p. 52) Compute the average dist. to center from a point in the circle of radius $R$

$X$ = dist to center. Recall $f(x) = \begin{cases} \frac{2x}{R^2}, & 0 \le x \le R \\ 0, & \text{elsewhere} \end{cases}$

$$E[X] = \int_0^R x \cdot \frac{2x}{R^2}\, dx = \frac{2}{R^2} \int_0^R x^2\, dx = \frac{2}{R^2} \frac{R^3}{3} = \frac{2}{3} R$$

$\frac{2}{3}R > \frac{1}{2} R$ !

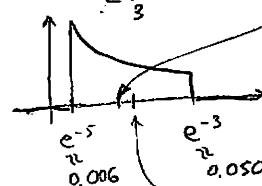


**Prop** (Expectation of a function of $X$). For any function $g(x)$,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

- Compare to $E[g(X)] = \sum_x g(x) f(x)$ for discrete r.v. $X$ (Prop 4.1 p. 33 of notes)
- **Ex** (refer to Ex. p. 54 on concentration of alcohol in blood): $X \sim \text{Unif}[3,5]$, $Y = e^{-X}$

$$E[Y] = \int_{-\infty}^{\infty} e^{-x} f_X(x)\,dx = \frac{1}{2}\int_3^5 e^{-x}\,dx = \frac{1}{2}(e^{-3} - e^{-5}) \approx 0.022$$




- **Ex** | $X \sim \text{Unif}[0,1]$. $Var(X) = ?$

$$Var(X) = E[X^2] - (\underbrace{E[X]}_{1/2})^2.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

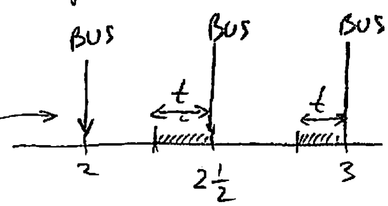
$$Var(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

Generally, $\boxed{\text{for } X \sim \text{Unif}[\alpha,\beta]: \quad E[X] = \frac{\alpha+\beta}{2}, \quad Var(X) = \frac{(\beta-\alpha)^2}{12}.}$ (Ex. 3a in Ross).

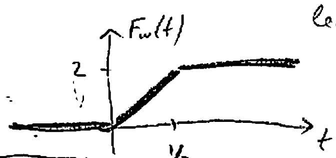
**Ex** | (Bus stop problem) Buses come to the bus stop at 1:00, 1:30, 2:00, 2:30 pm, etc.
Matt comes to the stop between 2 and 3 pm, uniformly distributed.
What is the distr. of waiting time? Expected wait? St. deviation?

$X$ = Matt's arrival $\sim \text{Unif}[2,3]$

$W$ = Matt's wait time




$$F_W(t) = P\{W \le t\} = P\{\underbrace{2\tfrac{1}{2} - t \le X \le 2\tfrac{1}{2}}_{\text{length} = t}\} + P\{\underbrace{3 - t \le X \le 3}_{\text{length} = t}\} = \frac{2t}{3-2} = \begin{cases} 2t, & 0 \le t \le \frac{1}{2} \\ 0, & t \le 0 \\ 1, & t \ge \frac{1}{2} \end{cases}$$




Hence $W \sim \text{Unif}[0, 1/2]$

$$EW = \frac{1}{4} = 15 \text{ min}$$

$$Var(W) = \frac{(1/2)^2}{12} \Rightarrow \sigma_W = \frac{1/2}{\sqrt{12}} \approx 0.12 \approx 7 \text{ min}.$$

So, wait time is $15 \pm 7$ min on average.