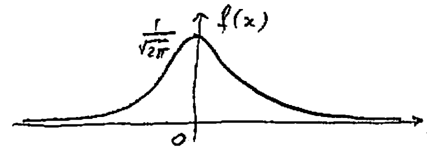
Standard Normal distribution (5.4)

Def A r.v. X has the standard normal distribution if the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty.$$



Notation: $X \sim N(0,1)$.

- Gauss used normal distr to model his observations in astronomy, so "normal" is often called "Gaussian"
- Why $\frac{1}{\sqrt{2\pi}}$?

Prop $f(x)$ above is indeed a pdf, i.e.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Proof Need to show:

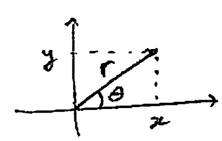
$$I := \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Trick: $$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right) = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy$$ (by Fubini Thm).

Pass to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta.$$

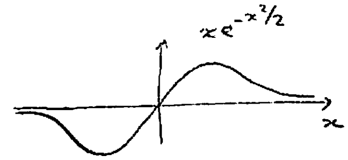


$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta = 2\pi \int_0^{\infty} r e^{-r^2/2}\,dr \underset{\substack{u = r^2/2 \\ du = r\,dr}}{=\!=\!=} 2\pi \int_0^{\infty} e^{-u}\,du = 2\pi.$$

Hence $I = \sqrt{2\pi}$. QED.

Prop For $X \sim N(0,1)$, $\boxed{E[X] = 0 \text{ and } Var(X) = 1}$ (Thus $\sigma_X = 1$)

Proof $E[X] = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \underbrace{x e^{-x^2/2}}_{\text{odd function}} dx = 0.$



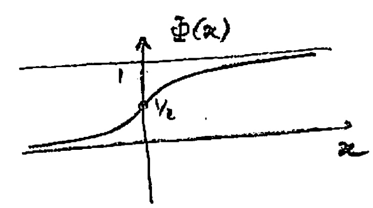
$Var(X) = E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx \;=$

(By parts: $u = x$, $dv = x e^{-x^2/2} dx \Rightarrow v = \int x e^{-x^2/2} dx = -e^{-x^2/2}$)

$= \frac{1}{\sqrt{2\pi}} \Big[\underbrace{-x e^{-x^2/2} \Big|_{-\infty}^{\infty}}_{= 0} + \int_{-\infty}^{\infty} e^{-x^2/2} dx \Big] = 1$ by Prop. p. 56. QED.

• **Cdf of $N(0,1)$** is denoted $\Phi(a)$; recall $\Phi(a) = P\{X \le a\}$

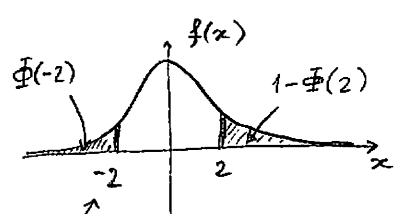
$$\boxed{\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx, \quad x \in \mathbb{R}}$$



$\Phi(x)$ can not be expressed in terms of elementary functions like $\sin x$, e^x, etc.
It is a "special function". Tabulated on p. 201 of Ross.

On a computer, can be computed as $\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$
"error function", another special function.

Ex Let $X \sim N(0,1)$. Compute
$P\{|X| \le 2\}$



$P\{-2 \le X \le 2\} = \Phi(2) - \Phi(-2)$?

By the symmetry of the pdf $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ (even function), we have
$\Phi(-2) = 1 - \Phi(2)$

$\Rightarrow P\{|X| \le 2\} = 2\Phi(2) - 1 \approx 0.954$.

• $\Rightarrow$ fast tail decay of $f(x)$. Ex $P\{|X| \le 3\} \approx 0.997$

• Remark: By the same symmetry reasoning,

$$\boxed{\Phi(-x) = 1 - \Phi(x)} \quad \text{for all } x \qquad (*)$$