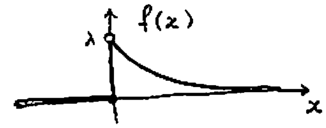
5.5 Exponential distribution

Def A r.v. X has the <u>exponential distribution</u> with parameter $\lambda > 0$ if the pdf of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



Notation:

$$X \sim \mathrm{Exp}(\lambda)$$

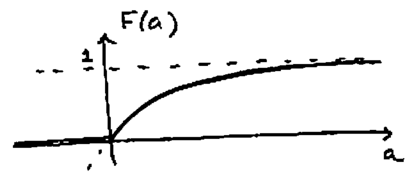
λ is called the <u>rate</u>

- Remark: $X \geq 0$ always (since $f(x)=0$ for $x<0$)

- $f(x)$ is <u>indeed a pdf</u>, since

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty} = 1$$

- <u>cdf</u>: $F(a) = \int_0^a \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_a^{\infty} = \begin{cases} 1 - e^{-\lambda a}, & a \geq 0 \\ 0, & a < 0 \end{cases}$



<u>Tails</u>: $P\{X > a\} = 1 - F(a) = e^{-\lambda a}$, $a \geq 0$ (exponential decay)

- <u>$E[X]$</u> $= \int_0^{\infty} x \lambda e^{-\lambda x}\,dx \overset{}{\underset{y=\lambda x}{=\!=}} \frac{1}{\lambda}\int_0^{\infty} y e^{-y}\,dy$ By parts: $\left\{ u = y, \; dv = e^{-y}dy \atop v = -e^{-y} \right\}$

$$= \frac{1}{\lambda}\Big[\underbrace{-ye^{-y}\Big|_0^{\infty}}_{=0} + \underbrace{\int_0^{\infty} e^{-y}\,dy}_{=1}\Big] = \boxed{\frac{1}{\lambda}}.$$

- <u>$\mathrm{Var}(X)$</u> $= ?$

$$E[X^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx = \frac{2}{\lambda^2} \quad \text{(by parts twice - Ex.)}$$

$$\Rightarrow \mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

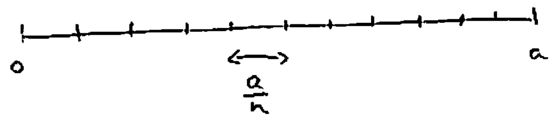
$$\sigma_X = \frac{1}{\lambda}.$$

WAITING TIME

- Exponential distr is a typical model for waiting time (continuous) until some event occurs. WHY?

Ex | X = time of the first call at a police station (say, after midnight). Let's find the distribution of X.

$P\{X \le a\} = ?$ Divide $[0,a]$ into n intervals of length $\frac{a}{n}$ (n large; e.g. every interval is a second)



$P\{\text{call during a given interval}\} = \lambda \cdot \frac{a}{n}$, for some coeff. λ (this prob. is proportional to the length of the interval)

$$\Rightarrow P\{X > a\} = P\{\text{no call in } [0,a]\} = P\{\text{no call during each interval}\}$$

$$= \left(1 - \frac{\lambda a}{n}\right)^n \quad \text{(by independence)}$$

$$= e^{-\lambda a} \quad \text{as } n \to \infty$$

Hence $\boxed{X \sim Exp(\lambda)}$

- **Prop** (Memoryless Property) Let $X \sim Exp(\lambda)$. Then

$$P\{X > t+s \mid X > t\} = P\{X > s\} \quad \text{for } s,t > 0$$

$P\{$need to wait $> s$ more minutes after having waited t min$\}$ $\qquad$ $P\{$need to wait $> s$ minutes$\}$

Proof: $\text{L.H.S} = \dfrac{P\{X > t+s\}}{P\{X > t\}} = \dfrac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \text{RHS}.$ QED.

THM Exponential distribution is the only continuous memoryless distribution

Applications: All of these have exponential distribution.

- time until next e-mail message arrives
- time until next customer enters the store
- lifetime of equipment (e.g. a cellphone) if there is no aging; can only die due to an accident
- time between successive independent events (accidents, e-mails, payments, etc.)

Ex | A small store is visited by 3 customers per hour, on average.
In the last hour there were no customers in the store.

(a) How many do we expect in the next hour?

Ans = 3 (memoryless)

(b) What is the prob. that in the next 20 min there will still be no customers?

X = time when the first customer enter the store

$X \sim Exp(\lambda)$. $E[X] = \frac{1}{\lambda} = \frac{1}{3}$, since ave. wait time for a customer is $\frac{1}{3}$ hour.

$\Rightarrow \underline{\lambda = 3}$.

$$P\{X > \underset{\substack{\uparrow \\ \text{30 min}}}{\tfrac{1}{3}}\} = e^{-\lambda \cdot 1/3} = e^{-1} \approx 0.37.$$

(c) What is the prob. that the next two customers will enter at most 5 min from each other?

Start counting time after the first customer enters.

Second customers is independent of the first $\Rightarrow$ his/her arrival time is $X \sim Exp(3)$.

$$P\{X < \underset{\substack{\uparrow \\ \text{5 min}}}{\tfrac{1}{12}}\} = 1 - e^{-3 \cdot \frac{1}{12}} = 0.22.$$

· Relation between <u>exponential</u> and <u>geometric</u> distributions:

- $Exp(\lambda)$ is a continuous version of $Geom(p)$

  $\uparrow$ pdf $= \lambda e^{-\lambda x}$ $\leftarrow$ exponential decay $\rightarrow$ $\uparrow$ pmf $= (1-p)^{n-1} p$

- $Exp(\lambda)$ is the only <u>continuous</u> memoryless distr;
  $Geom(p)$ is the only <u>discrete</u> memoryless distr.

- If $X \sim Exp$, then $\lceil X \rceil \sim Geom$.