6.1. Joint distributions

Now we study pairs of r.v's X, Y. Study relations between X, Y.

Examples: (a) X = person's height, Y = person's weight

(b) X = person's total cholesterol; Y = # hours the person exercises per week.

(c) X = price of an item; Y = # items sold

etc

(a) Discrete distributions

Def • X, Y : discrete r.v's. The joint pmf of X, Y is

$$p(x,y) = P\{X=x,\ Y=y\}.$$

• The individual pmf's $p_X(x) = P\{X=x\}$, $p_Y(y) = P\{Y=y\}$ are called the marginal pmf's.

Remarks (a) (marginals from joint):

$$p_X(x) = \sum_y P\{X=x,\ Y=y\} = \sum_y p(x,y)$$

$$p_Y(y) = \sum_x P\{X=x,\ Y=y\} = \sum_x p(x,y).$$

(b) joint from marginals — impossible.

Ex Results of a survey of 100 apartments.

Y = # iPads

X = # bedrooms

X \ Y	0	1	2
0	5	12	3
1	7	10	8
2	2	26	27

⇒ joint pmf $p(x,y)$:

X \ Y	0	1	2	Row totals give marginal $p_X(x)$
0	0.05	0.12	0.03	0.2
1	0.07	0.1	0.08	0.25
2	0.02	0.26	0.27	0.55
Column totals give marginal $p_Y(y)$	0.14	0.48	0.38	GREAT TOTAL IS 1

Prob{a random apt has 1 bdrm & 1 iPad}

Prob{a random apt has 1 iPad}

Prob{a random apt is 1 bdr}

Ex Roll a pair of dice.

X = the larger number, Y = the smaller

Compute the joint and marginal distributions.

$p(1,1) = p(2,2) = \cdots = p(6,6) = \frac{1}{6}\cdot\frac{1}{6} = 36$

If $x > y$, $\quad p(x,y) = P\{(xy) \text{ or } (yx) \text{ are rolled}\} = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}$.

$y \backslash x$	1	2	3	4	5	6	$p_Y(y)$ ← marginal
6	0	0	0	0	0	1/36	1/36
5	0	0	0	0	1/36	1/18	3/36
4	0	0	0	1/36	1/18	1/18	5/36
3	0	0	1/36	1/18	1/18	1/18	7/36
2	0	1/36	1/18	1/18	1/18	1/18	9/36
1	1/36	1/18	1/18	1/18	1/18	1/18	11/36
$p_X(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	

↑ marginal

__Note__: X, Y are __not "uniform"__ on $\{1, 2, \ldots 6\}$.

(b) Continuous distributions

__Def__. We say that X, Y have a __continuous joint distribution__ if there exists a function $f(x,y) \geq 0$ defined on $\mathbb{R}\times\mathbb{R}$ and s.t.

$$P\{(X,Y)\in C\} = \iint_C f(x,y)\,dx\,dy \quad \text{for all } C \subset \mathbb{R}^2$$

- $f(x,y)$ is called the __joint pdf__ of X, Y.
- The individual pdf's $f_X(x)$ and $f_Y(y)$ are called the __marginal pdf's__.

__Prop__ (__Marginals from joint__)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy,$$

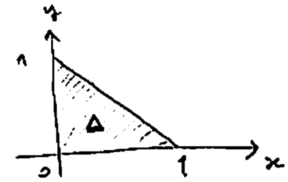
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

__Proof__: $P\{X\in A\} = P\{X\in A,\ Y\in(-\infty,\infty)\} = \int_A \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}\,dx$

$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dx$ by def. of pdf. QED.

Ex Let (X,Y) have joint pdf

$$f(x,y) = \begin{cases} kxy & \text{if } x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$



Compute k; the marginal pdf's. of X, Y.

Remark: $\underbrace{\text{"Repulsion"}}_{xy \text{ forces}}$ of (X,Y) from x-axis, y-axis.

k: $\iint_{-\infty}^{\infty} f(x,y) = 1 \qquad \iint_{\Delta} kxy\,dx\,dy = 1 \quad \Rightarrow \quad k = \dfrac{1}{\iint_{\Delta} xy\,dx\,dy}$

$$\iint_{\Delta} xy\,dx\,dy = \int_0^1 dx \int_0^{1-x} xy\,dy = \int_0^1 dx \int_0^{1-x} y\,dy$$

$$= \int_0^1 \frac{x(1-x)^2}{2}\,dx = \frac{1}{2}\int_0^1 (x^3 - 2x^2 + x)\,dx = \frac{1}{2}\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) = \frac{1}{24} \Rightarrow k = 24$$

Marginals: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} 24xy = 24x\int_0^{1-x} y\,dy = \dfrac{24x(1-x)^2}{2} = \begin{cases} 12x(1-x)^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$

Similarly, $f_Y(y) = \begin{cases} 12y(1-y)^2, & 0 \leq y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$