lec 25 03/12

6.2 Independent r.v's

Def R.v's X, Y are <u>independent</u> if
$$P\{X\in A, Y\in B\} = P\{X\in A\}P\{Y\in B\} \quad \forall A, B \subset \mathbb{R}.$$
In other words ~~P{X∈A}~~ events $\{X\in A\}$ and $\{Y\in B\}$ are indep.

Let X, Y indep.

~~CDF~~

PMF: $P\{X=x, Y=y\} = P\{X=x\}P\{Y=y\}$
$$\boxed{p(x,y) = p_X(x)\,p_Y(y)} \qquad (*)$$

PDF:
$$\boxed{f(x,y) = f_X(x)\,f_Y(y)} \qquad (**)$$

THM X, Y are indep $\Leftrightarrow$ $(*)$ ~~or~~ $(**)$ holds.

Ex X = larger of the two dice, Y = smaller. (p. 53)
<u>Not</u> independent! ~~(Check)~~ (see PMF)

Ex (a) $f(x,y) = kxy$, $(x,y)\in D$ (p. 54) –
<u>not</u> independent (see PDF).

(b) $f(x,y) = kxy$, $0<x<1$, $0<y<1$ – <u>independent</u>.

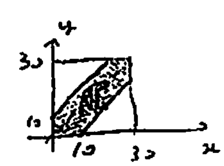
Ex Two people ~~decided to~~ agreed to meet in a lobby between 11:30 and 12 noon.
Each arrives at random during this period.
What is the prob. that ~~they meet~~ one waits for the other at most 10 min?

($10/30 = 1/3$ – is wrong!) Wait time is <u>not</u> uniform $(0,30)$

$X, Y \sim$ Uniform $(0,30)$, independent.

$f_X(x) = \frac{1}{30}$, $x\in(0,30)$
$f_Y(y) = \frac{1}{30}$, $y\in(0,30)$

$\Rightarrow f(x,y) = f_X(x)f_Y(y) = \frac{1}{900}$, $(x,y)\in[0,30]^2$
<u>Uniform</u> on $[0,30]^2$



$$P\{|X-Y|\le 10\} = \iint_C f(x,y)\,dx\,dy = \iint_C \frac{1}{900}\,dx\,dy = \frac{\text{Area}(C)}{900} = \frac{500}{900} = \boxed{\frac{5}{9}} \quad (\ge \tfrac{1}{2}).$$
← Area (square)

Geometric probability

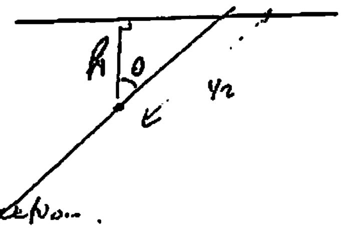
Ex Compute the distribution (cdf, pdf) of wait time.

Fact If $(X,Y)\sim$ Uniform (S), then $P\{(X,Y)\in C\} = \frac{\text{Area}(C\cap S)}{\text{Area}(S)}$

Ex (Buffon's needle) (18 century)

Drop a needle of length 1 onto a ruled paper with dist. 1 between lines
Compute the probability of intersection.

~~Let X = dist to the nearest~~

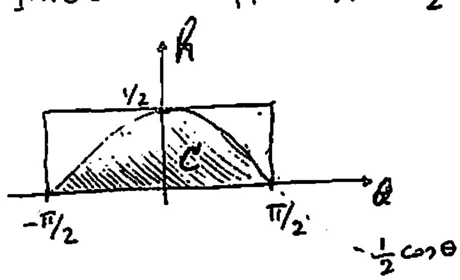


h = dist from the center of needle to the nearest line.

θ = angle of needle with the orthog direction.

Then: (model) $h \sim \text{Unif}(0, \frac{1}{2})$, $\quad \theta \sim \text{Unif}(-\frac{\pi}{2}, \frac{\pi}{2})$, independent

Intersection iff $\quad h \le \frac{1}{2}\cos\theta$.



By geometric probability,

$$P\{h \le \tfrac{1}{2}\cos\theta\} = \frac{\text{Area}(C)}{\frac{1}{2}\cdot\pi} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos\theta \, d\theta$$

$$= \frac{1}{\pi}\sin\theta\Big|_{-\pi/2}^{\pi/2} = \boxed{\frac{2}{\pi}}$$

Implications: can compute π by repeating.

~~Ex (Bivariate normal). X, Y ~ N(0,1) independent. Joint PDF?~~

~~$f(x,y) = f(x) f(y) =$~~

Def (Multivariate) $X_1, X_2 \ldots X_n$ indep. if
$$P(X_1 \in A_1, \ldots, X_n \in A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n).$$

PMF: $\quad p(x_1, x_2 \ldots x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$

PDF: $\quad f(x_1 \ldots x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

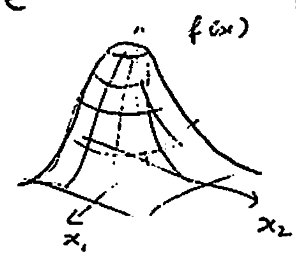
Ex (Multivariate normal) $\quad X_1, \ldots, X_n \sim N(0,1)$ independent.

Joint PDF $\quad f(x_1, \ldots, x_n) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \cdots \frac{1}{\sqrt{2\pi}} e^{-x_n^2/2} = \frac{1}{(2\pi)^{n/2}} e^{-(x_1^2 + \cdots + x_n^2)/2}$

$x = (x_1 \ldots x_n)$. $\quad |x| = \sqrt{x_1^2 + \cdots + x_n^2}$

$$\boxed{f(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}, \quad x \in \mathbb{R}^n.}$$

$N(0, I)$



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