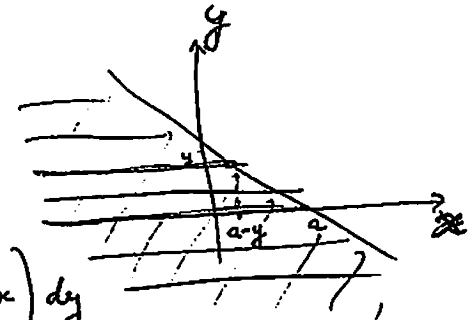
- Sums of general indep. r.v's. X, Y.



CDF: $F_{X+Y}(a) = P\{X+Y \le a\} = \iint_{(x,y)\in\mathbb{R}^2:\ x+y\le a} \underbrace{f_{X,Y}(x,y)}_{f_X(x)f_Y(y) \text{ by indep}}\,dx\,dy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)\, f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty} f_Y(y) \underbrace{\left(\int_{-\infty}^{a-y} f_X(x)\,dx\right)}_{F_X(a-y)} dy$

$= \int_{-\infty}^{\infty} F_X(a-y)\, f_Y(y)\,dy.$

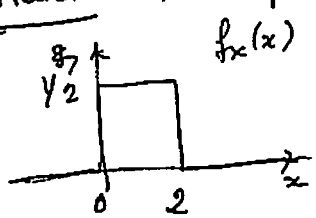
Differentiate w.r.t. to a.

PDF: $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\,dy \;=: (f_X * f_Y)(a).$

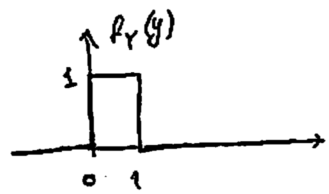
PROP X, Y indep. continuous. $\Rightarrow$ the PDF of $X+Y$ is given by the convolution of PDF's of X, Y;

$$f_{X+Y}(a) = (f_X * f_Y)(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\,dy.$$

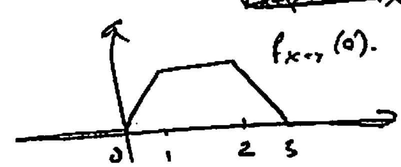
- Rework Ex. on p. 59 in terms of convolutions!



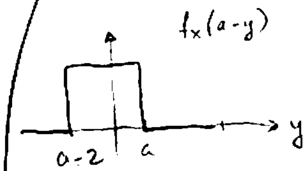
$*$



$=$



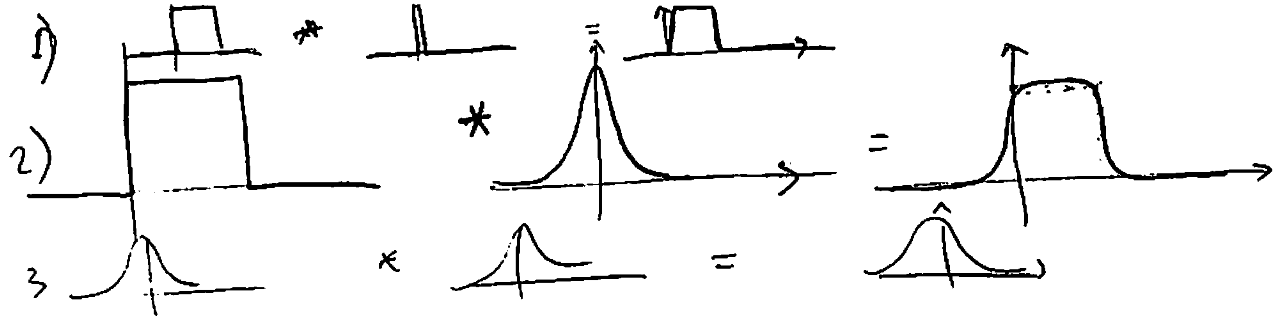
Check for $0 < a < 1$: $f_{X+Y}(a) = \int_{-\infty}^{\infty} \underbrace{f_X(a-y)\, f_Y(y)}_{\begin{cases} 1/2, & a-y\in(0,2),\ y\in(0,1) \\ 0, \end{cases}} dy$



$= \begin{cases} 1/2, & 0 < y < a \\ 0, \end{cases}$

$= \int_0^a \frac{1}{2}\,dy = a/2.$

Ex



- Sums of independent normals is normal

Ex ~~X~N(0,~~ ~~X~~ $X, Y \sim N(0,1)$ indep. $X+Y \sim$?

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f(a-y)\, f(y)\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(a-y)^2/2}\, e^{-y^2/2}\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(y^2 - ay + a^2/2)}\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(y-\frac{a}{2}\right)^2 + \frac{a^2}{4}\right]}\, dy = e^{-a^2/4}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(y-\frac{a}{2}\right)^2}\, dy$$

$\left(y-\frac{a}{2}\right)^2 = \frac{x^2}{2}$, $y - \frac{a}{2} = \frac{x}{\sqrt{2}}$

$$= e^{-a^2/4}\, \frac{1}{2\sqrt{2}\cdot\pi} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)}_{=1} \cdot \frac{1}{2\sqrt{\pi}}\, e^{-a^2/4}.$$

$$= \frac{1}{2\sqrt{\pi}}\, e^{-a^2/4} = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-a^2/2\sigma^2} \quad \text{for } \sigma = \sqrt{2}.$$

$$\boxed{X + Y \sim N(0, 2)}$$

More generally:

THM $X_i \sim N(\mu_i, \sigma_i)$ indep. $\Rightarrow \sum_{i=1}^{n} X_i \sim N(\mu, \sigma)$ where

$$\mu = \sum_{i=1}^{n} \mu_i, \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

(mean) (variance)

In particular, if $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1 \Rightarrow \sigma = \sqrt{2}$.

Lec 24 03/09

Ex [Ghahramani Ex 11.9] Class = 25 students

Students' grades in an exam are normally distributed with mean 72 and variance 15

Prob { average grade of this room ~~course~~ ~~in a class of~~ ≥ 74 } = ?

$X_1, \dots, X_{25}$ = grades, $X_k \sim N(\mu, \sigma^2)$, $\mu = 72$, $\sigma^2 = 15$.

Ave. $\bar{X} = \frac{1}{25}\sum_{k=1}^{25} X_k \sim N\left(\frac{1}{n}\sum_1^n \mu, \ \frac{1}{n^2}\sum_1^n \sigma^2\right)$ by THM p. 67

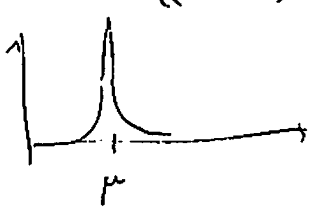
$= N\left(\mu, \frac{\sigma^2}{n}\right)$

$= N(72, 0.6)$

$1 - \Phi(2.58) \approx 0.0049$

$P\{\bar{X} > 74\} = P\left\{\frac{\bar{X} - 72}{\sqrt{0.6}} > \underbrace{\frac{74 - 72}{\sqrt{0.6}}}_{2.58}\right\}$ ~~$= P\{\bar{X} \dots\} = 1 - \Phi(3) \approx 0.0013$~~

Remark on $N\left(\mu, \frac{\sigma^2}{n}\right)$: Variance small $\Rightarrow$ the bigger the sample, ~~the better the~~ the tighter the estimate of μ by $\bar{X}$



Sums of independent discrete r.v's.

THM. 1. Let $X_1, \dots, X_n \sim$ Bernoulli(p) indep. Then $X_1 + \dots + X_n \sim$ Binom(n, p)

2. Let $X \sim$ Binom(n, p), $Y \sim$ Binom(m, p) indep. Then $X + Y \sim$ Binom$(n+m, p)$.

3. Let $X \sim$ Poisson(λ), $Y \sim$ Poisson(μ). Then $X + Y \sim$ Poisson$(\lambda + \mu)$.

1, 2 – explain in terms of successes / trials.

3: Express $\lambda = np$, $\mu = mp$ where $n, m \to \infty$, $p \to 0$. Then ~~approx~~ in the limit,

$X \sim$ Binom(n, p), $Y \sim$ Binom(m, p).

By part 2, $X + Y \sim$ Binom$(n+m, p) \approx$ Poisson$((n+m)p)$ in the limit
$\sim$ Poisson$(\lambda + \mu)$