Lec 24 03/09

Ex [Ghahramani Ex 11.4] Class = 25 students

Students' grades in an exam are normally distributed with mean 72 and variance 25

Prob{average grade of this course ~~in a class of~~ ≥ 75} = ?

$X_1, \dots, X_n$ = grades, $X_k \sim N(\mu, \sigma^2)$, $\mu = 72$, $\sigma = \sqrt{25} = 5$.

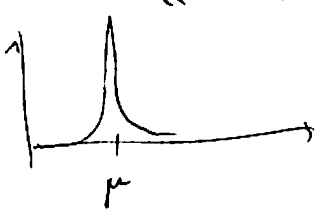
Ave. $\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k \sim N\left(\frac{1}{n} \sum_1^n \mu, \ \frac{1}{n^2} \sum_1^n \sigma^2\right)$ by THM p.61

$$= N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$= N(72, 1)$$

$$P\{\bar{X} > 75\} = P\left\{\frac{\bar{X} - 72}{1} > \underbrace{\frac{75-72}{1}}_{=3}\right\} = 1 - \Phi(3) \approx 0.0013$$

Remark on $N\left(\mu, \frac{\sigma^2}{n}\right)$: Variance small $\Rightarrow$ the bigger the sample, ~~the better the~~ the tighter the estimate of μ by $\bar{X}$



Sums of independent discrete r.v's.

THM. 1. Let $X_1, \dots, X_n \sim$ Bernoulli(p) indep.
Then $X_1 + \dots + X_n \sim$ Binom(n,p)

2. Let $X \sim$ Binom(n,p), $Y \sim$ Binom(m,p) indep. Then $X + Y \sim$ Binom$(n+m, p)$.

3. Let $X \sim$ Poisson(λ), $Y \sim$ Poisson(μ). Then $X + Y \sim$ Poisson$(\lambda + \mu)$.

1, 2 – explain in terms of successes / trials.

3: Express $\lambda = np$, $\mu = mp$ where $n, m \to \infty$, $p \to 0$. Then ~~as~~ in the limit,
$X \sim$ Binom(n,p), $Y \sim$ Binom(m,p).
By part 2, $X + Y \sim$ Binom$(n+m, p) \approx$ Poisson$((n+m)p)$ in the limit
$\sim$ Poisson$(\lambda + \mu)$

Ex The number of male customers in the store is ~ Poisson with mean 5
female — — — — — 10.

The ~~num~~ total # of customers ?

~~X~~ Male: $X \sim \text{Poisson}(5)$
Female: $Y \sim \text{Poisson}(10)$
$X+Y \sim \text{Poisson}(15)$.

Sums of continuous r. v's

~~Ex Calls at a police station after midnight ~ Exp(λ).~~

Ex Interval btw calls at a police station is $\sim \text{Exp}(\lambda)$ (thus, ave $= 1/\lambda$)
Distr. of the second call after midnight?

(Another example: time to service a car $\sim \text{Exp}(\lambda)$. Distr. of time to service 2 cars ?)

(first call $\sim \text{Exp}(\lambda)$).

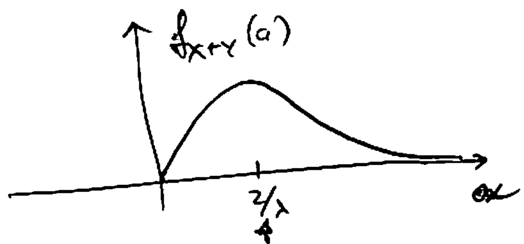
Recall pdf of $\text{Exp}(\lambda)$. $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$X, Y \sim \text{Exp}(\lambda)$. $X+Y \sim ?$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f(a-y) f(y)\,dy = \int_0^a \lambda e^{-\lambda(a-y)} \lambda e^{-\lambda y}\,dy$$

$$= \lambda^2 e^{-\lambda a} \int_0^a dy = \lambda^2 a e^{-\lambda a}$$

$$= \begin{cases} \underbrace{\lambda e^{-\lambda a}}_{\text{exp pdf}} \cdot \underbrace{\lambda a}_{\text{correction for 2'nd call}}, & a > 0 \\ 0, & a < 0. \end{cases}$$



Remark: $E[X+Y] = 2/\lambda$.

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(or: time to service n cars)

Ex Distr. of n'th call is proportional to $\lambda e^{-\lambda a}\cdot(\lambda a)^{n-1}$, $a>0$ (**Check!**)

$$\Rightarrow \quad f_n(x) = \frac{1}{C}\cdot \lambda e^{-\lambda a}(\lambda a)^{n-1}.$$

$$\int_0^\infty f_n(x)\,dx = 1 \quad \Rightarrow \quad C = \int_0^\infty \lambda e^{-\lambda a}(\underbrace{\lambda a}_{=t})^{n-1} = \boxed{\int_0^\infty e^{-t}t^{n-1}\,dt = (n-1)!}$$

see Wikipedia

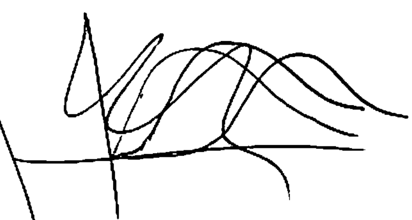
$$\boxed{f_n(x) = \frac{\lambda e^{-\lambda a}(\lambda a)^{n-1}}{(n-1)!}}, \quad a>0$$

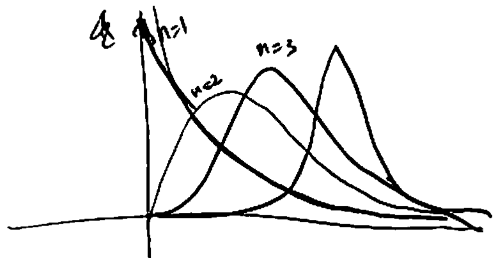
- Extension from $n\in\mathbb{N}$ to $n\in\mathbb{R}$: this integral $= \Gamma(n)$ (def of Gamma function)
  $\Gamma(n) = (n-1)!$ for $n\in\mathbb{N}$.

Gamma distribution:

Def $X \sim Gamma(n,\lambda)$ if PDF

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{\Gamma(n)}, & x\ge 0 \\ 0, & x<0 \end{cases}$$





So, n'th call $\sim Gamma(n,\lambda)$. Formally: ~~$X,Y\sim Exp(\lambda)$ indep $\Rightarrow X+Y\sim Gamma(2,\lambda)$~~

$n=1$: $Exp(\lambda) \sim Gamma(1,\lambda)$.

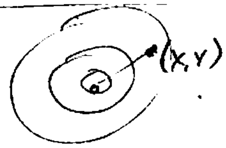
THM 1. $X_1,\dots,X_n \sim Exp(\lambda)$ indep $\Rightarrow X_1+\dots+X_n \sim Gamma(n,\lambda)$.
2. $X_1\dots X_n \sim Gamma(n_i,\lambda) \Rightarrow X_1+\dots+X_n \sim Gamma(\sum n_i,\lambda)$.

⌈1. already proved

2. For $n_i\in\mathbb{N}$, ~~$X_1+\dots+X_n$ is the time of $(\sum_1 n_i)$-th call $\Rightarrow$ Gamma~~ each X_i is a sum of n_i exponentials $\Rightarrow$ $\sum X_i$ is the sum of $\sum n_i$ exponentials $\Rightarrow$ finish by part 1 ⌋

Prop $X\sim Gamma(n,\lambda) \Rightarrow E[X] = \frac{n}{\lambda}$, $Var(X) = n/\lambda^2$.

(by Thm + prop. of exp.)

Ex. Shooting at Target  (X,Y). The shot (X,Y): $X, Y \sim N(0,1)$ indep.

Distr. of $R^2 = X^2 + Y^2$?

Distr. of X^2 ?

$$F_{X^2}(a) = P\{X^2 \le a\} = P\{-\sqrt{a} \le X \le \sqrt{a}\} = \Phi(\sqrt{a}) - \Phi(-\sqrt{a})$$

$$= 2\Phi(\sqrt{a}) - 1$$

$$\Rightarrow \quad f_{X^2}(a) = \frac{d}{da} F_{X^2}(a) = 2 \frac{d}{da} \Phi(\sqrt{a})$$

$$= 2 f_X(\sqrt{a}) \cdot \frac{1}{2\sqrt{a}} \qquad \text{where } f_X(\cdot) \text{ is the PDF of } N(0,1)$$

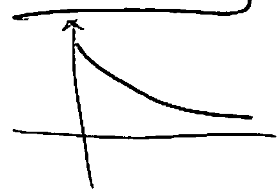
$$= \frac{1}{\sqrt{a}} f_X(\sqrt{a}) = \frac{1}{\sqrt{a}} \frac{1}{\sqrt{2\pi}} e^{-a/2} \qquad (= \text{Gamma?})$$

$$= \frac{1}{2} e^{-a/2} (a/2)^{\frac{1}{2}-1} \cdot C$$

$$\Rightarrow \quad X^2 \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

$$X^2 + Y^2 \sim \text{Gamma}\left(\tfrac{1}{2}+\tfrac{1}{2}, \tfrac{1}{2}\right) = \text{Gamma}\left(1, \tfrac{1}{2}\right) \sim \boxed{\text{Exp}\left(\tfrac{1}{2}\right)}$$

$$f_{X^2+Y^2}(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \ge 0 \\ 0, & \end{cases}$$



~~In particular, $E[R^2] = 2$.~~

Remark More generally, if $X_1, \dots, X_n \sim N(0,1)$ then

$$X_1^2 + \dots + X_n^2 \sim \text{Gamma}\left(\tfrac{n}{2}, \tfrac{1}{2}\right) =: \boxed{\chi_n^2}$$

Chi-square distr. with n degrees of freedom.

Summary: $\sum$ Bernoulli = Binomial, $\sum$ Binom = Binom, $\sum$ Poisson = Poisson

$\sum$ Normal = Normal, $\sum$ Exp = Gamma, $\sum$ Gamma = Gamma, $\sum \text{Normal}^2 = \chi^2$.

$\sum$ chi-square = chi-square.