~~lec 26 03/19~~ Lec 29 03/21

6.4. ~~Contr~~ Conditional distributions. Discrete case.

- Recall conditional prob: $P(E|F) = \dfrac{P(E\cap F)}{P(F)}$.

Conditional distr of X given ~~a value of Y~~ that $Y=y$ is the knowledge of

$$P\{X=x \mid Y=y\}; \quad \cancel{\text{for all } x} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}} = \frac{p(x,y)}{p_Y(y)} \begin{matrix} \leftarrow \text{joint} \\ \leftarrow \text{marginal} \end{matrix}$$

Def (Condit. distr.: discrete). X, Y r.v's.

- Conditional PMF

$$p_{X|Y}(x|y) = P\{X=x \mid Y=y\} = \frac{p(x,y)}{p_Y(y)}.$$

- Conditional CDF:

$$F_{X|Y}(x|y) = P\{X \le x \mid Y=y\}.$$

Remark: 1. If X, Y are independent then the conditional distr = unconditional:

$$p_{X|Y}(x|y) = \frac{p_X(x)\, p_Y(y)}{p_Y(y)} = p_X(x).$$

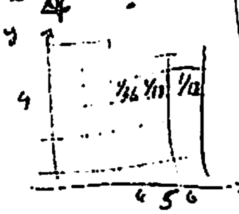
~~2. E~~

Def (Conditional expectation) (sect. 7.5.1)

$$E[X|Y=y] = \sum_x x\, P\{X=x \mid Y=y\} = \sum_x x\, p_{X|Y}(x|y).$$

Ex. (two dice: X = larger, Y = smaller) [lec. 24 03/09 p. 63]

- Conditional distr. of X given that $Y=4$?



$X \in \{4,5,6\}$

$$p_Y(4) = \frac{1}{36} + \frac{1}{18} + \frac{1}{18} = \frac{5}{36}$$

$$p_{X|Y}(x|4) = \begin{cases} \frac{1}{36} / \frac{5}{36} = \frac{1}{5}, & x=4 \\ \frac{1}{18} / \frac{5}{36} = \frac{2}{5}, & x=5 \\ \frac{1}{18} / \frac{5}{36} = \frac{2}{5}, & x=6 \end{cases}$$

- $E[X|Y=4] = \cancel{\frac{1}{3}\left(\frac{1}{5}+\frac{2}{5}+\frac{2}{5}\right) = \frac{1}{3}}$ $\quad \frac{1}{5}\cdot 4 + \frac{2}{5}\cdot 5 + \frac{2}{5}\cdot 6 = \boxed{5.2}$

~~66~~

Ex on X = # bedrooms, Y = # [illegible]

$p(x,y)$.

$x \backslash y$	0	1	2
0	0.05	0.12	0.03
1	0.07	0.1	0.08
2	0.02	0.26	0.71

Condit. distr. of X given $Y=2$.

$$p_{X|Y}(x|2) = \frac{p(x,2)}{\underbrace{p_Y(2)}_{=0.38}} = \begin{cases} 0.08, & x=0 \\ 0.21, & x=1 \\ 0.71, & x=2 \end{cases}$$

$E[X|Y=2] = 0\cdot 0.08 + 1\cdot 0.21 + 2\cdot 0.71 = 1.63$

Compare to $E[X] = 1.35$ (see p. 65)

Ex (~~The box~~ Exchange Paradox). [Ghahramani p. 347]

"The envelopes problem"

There are two identical ~~boxes~~ closed ~~boxes,~~ envelopes, ~~each~~ one containing twice as much money as the other.

You are asked to choose one of the ~~boxes~~ envelopes for yourself.

You pick an ~~box~~ envelope at random, ~~and~~ open it, observe the content.

You are given an option to ~~see~~ exchange it for the other ~~box~~ envelope.

Should you exchange or not?

Y = amount $ in the box you pick, X = in the other box

~~$E\{Y \mid X = x\}$~~ Conditional distr. of X given that $Y = y$.

$$P\{X = 2y \mid Y = y\} = P\{X = \tfrac{y}{2} \mid Y = y\} = \tfrac{1}{2}.$$

$$E\{X \mid Y = y\} = 2y \cdot \tfrac{1}{2} + \tfrac{y}{2} \cdot \tfrac{1}{2} = 1.25\, y. \quad (> x)$$

Hence should always exchange!

"It is always better to switch to the ~~other~~ second box, no matter how much money you find in the first one."

Solution of paradox: with these assumptions, ~~it would follow that~~

$E[Y] = \infty$ $ (∞ amount of money).

~~If E[X]~~ Indeed, if $E[Y] < \infty$ then by observing a large amount y, ~~I should not~~ I would ~~know the~~ conclude that the other box can not have $2y$ with prob $1/2$ (but rather with a smaller prob)

$\Rightarrow$ should not swap

§ 6.5 Conditional distr's : ~~di~~ Continuous case

Def (Cond. distr: continuous). X, Y : r.v's.

Conditional PDF :

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

~~Ex~~ **Remark** If X, Y are indep. then $f_{X|Y}(x|y) = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$.

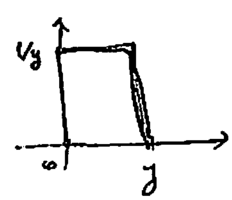
Def (Conditional expectation) (Sect. 7.5.1)

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\, dx.$$

Ex One breaks a stick of length 1 at a uniform random point.
Then breaks one of the pieces again.
Distr. of the remaining piece ?

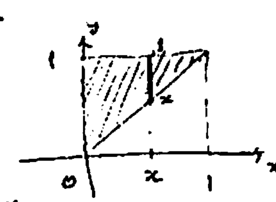
$Y \sim \mathrm{Unif}(0,1)$, $\quad X|Y \sim \mathrm{Unif}(0,Y)$. $\quad$ Distr. of X ?

~~[illegible]~~ • We know : $f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{---} \end{cases}$



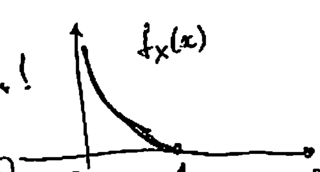
$$f_{X|Y}(x|y) = \begin{cases} 1/y, & 0 \le x \le y \\ 0, & \text{---} \end{cases}$$

• Joint distr. ~~pdf~~ : $f(x,y) = f_{X|Y}(x|y) f_Y(y) = \begin{cases} 1/y, & 0 \le x \le y \le 1 \\ 0, & \text{---} \end{cases}$



• Marginal distr. $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_x^1 \frac{1}{y}\, dy = \ln y \Big|_x^1 = -\ln x,$ $\quad 0 \le x \le 1$.

$$f_X(x) = \begin{cases} -\ln x, & 0 \le x \le 1 \\ 0, & \text{---} \end{cases}$$ Not uniform!



Consequence : $E[X] = -\int_0^1 x \ln x \, dx = \left(\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x\right)\Big|_0^1 = \frac{1}{4}$ ← heuristics : $\frac{1}{2} \times \frac{1}{2}$

Ex: $E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_0^y \frac{x}{y}\, dx = \frac{1}{y} \cdot \frac{y^2}{2} = \boxed{y/2}$. This is clear si[illegible]