\[(x, y) \sim \text{Unif}(C) \text{ where } C = \{ (x, y) : x^2 + y^2 \leq R^2 \} \]

1) Marginal distr. \( f_x(x), f_y(y) \)?

2) Are \( X, Y \) independent?

3) Conditional distr. \( f_{X|Y}(x|y) \)?

Joint pdf:
\[
f(x, y) = \begin{cases} \frac{1}{\text{Area}(C)} = \frac{1}{\pi R^2} & , \quad x^2 + y^2 \leq R \\
0 & , \quad \text{elsewhere},
\end{cases}
\]

1) \( f_x(x) = \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{-\infty}^{\infty} \frac{1}{\pi R^2} \, dy = \left\{ \begin{array}{ll}
\frac{2\sqrt{R^2 - x^2}}{\pi R^2}, & -R < x < R \\
0, & \text{elsewhere}
\end{array} \right. \)

2) No, since \( f(x, y) \neq f_x(x) \cdot f_y(y) \)

3) \( f_{X|Y}(x|y) = \frac{f(x, y)}{f_y(y)} = \frac{1}{\pi R^2} \cdot \frac{\pi R^2}{2\sqrt{R^2 - y^2}} \)

\[
= \left\{ \begin{array}{ll}
\frac{1}{2\sqrt{R^2 - y^2}}, & -\sqrt{R^2 - y^2} \leq x \leq \sqrt{R^2 - y^2} \\
0, & \text{elsewhere}
\end{array} \right. \]

\( X|Y \sim \text{Unif}\left(-\sqrt{R^2 - y^2}, \sqrt{R^2 - y^2}\right) \)
Refer to Ex. from 03/21.

\( Y \sim \text{Unif}(0,1), \, X|Y \sim \text{Unif}(0,Y) \)

\[ Y|X \sim? \quad E[Y|X=x] = ? \quad (\text{Prediction of } Y \text{ from } X=x) \]

\[
f_{Y|X}(y|x) = \begin{cases} 
\frac{f(x,y)}{f_X(x)} & \quad 0 \leq y \leq x \\
0 & \quad \text{elsewhere}
\end{cases}
\]

We already computed \( f_X(x) = \begin{cases} 
-x, & \quad 0 < x < 1 \\
0 & \quad \text{elsewhere}
\end{cases} \)

\[
f_{Y|X}(y|x) = \begin{cases} 
-\frac{1}{y|x}, & \quad 0 < y < x \\
0, & \quad \text{elsewhere}
\end{cases}
\]

\[
E[Y|X=x] = \int_0^x y \cdot f_{Y|X}(y|x) \, dy = \int_0^x -\frac{y}{y|x} \, dy = \begin{cases} 
\frac{1-x}{-\ln x}, & \quad 0 < x < 1 \\
0, & \quad \text{elsewhere}
\end{cases}
\]

For example: \( Y = 0.39, \quad X = 0.12 \quad \Rightarrow \text{our prediction of } Y \text{ is } \frac{1-0.12}{-\ln 0.12} \approx 0.42 \).

Continuous version of Bayes formula:

\[
f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y) \cdot f(y)}{\int_0^\infty f_{X|Y}(x|y) \cdot f(y) \, dy}
\]

\( f_Y = \text{"prior", } f_{Y|X} = \text{"posterior" distributions of } Y \)

Before / after getting information about the value of \( X \).

In our example, \( f_{Y|X}(y|x) = \begin{cases} 
\frac{1}{yx}, & \quad 0 < y < x \\
0, & \quad \text{elsewhere}
\end{cases} \) (same answer)

\[ \text{Ex. The lifetime of a bulb is } \text{Exp}(\lambda) \]

\( \lambda \sim \text{Unif}[0,1] \). Find the (marginal) dist. of the lifetime of the bulb.

\[ X \sim \text{Exp}(\lambda), \quad f_{X|\lambda}(x|\lambda) = \begin{cases} 
e^{-\lambda x}, & \quad x > 0 \\
0 & \quad \text{elsewhere} \end{cases}, \quad f_{\lambda}(\lambda) = \frac{1}{\beta}, \quad 0 < \lambda \leq \beta. \]

\[
f(x, \lambda) = f_{X|\lambda}(x|\lambda) \cdot f_{\lambda}(\lambda) = \begin{cases} 
\frac{\lambda}{\beta} e^{-\lambda x}, & \quad x > 0 \\
0, & \quad \text{elsewhere}
\end{cases}
\]

\[
f(x) = \int_0^\beta f(x, \lambda) \, d\lambda = \frac{e^{-\lambda x} - e^{-(\beta + \lambda) x}}{(\beta - \alpha) x^2}
\]
Simpson's paradox

From Wikipedia, the free encyclopedia

In probability and statistics, Simpson's paradox (or the Yule-Simpson effect) is a paradox in which a correlation (trend) present in different groups is reversed when the groups are combined. This result is often encountered in social-science and medical-science statistics, and it occurs when frequency data are hastily given causal interpretations.

<table>
<thead>
<tr>
<th></th>
<th>Applicants</th>
<th>Admitted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>8442</td>
<td>44%</td>
</tr>
<tr>
<td>Women</td>
<td>4321</td>
<td>35%</td>
</tr>
</tbody>
</table>

Marginal distribution of $X$ is skewed toward $X=1$

However when examining the individual departments, it was found that no department was significantly biased against women. In fact, most departments had a "small but statistically significant bias in favor of women".

<table>
<thead>
<tr>
<th>Department</th>
<th>Men ($X=1$)</th>
<th>Women ($X=2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Applicants</td>
<td>Admitted</td>
</tr>
<tr>
<td>1</td>
<td>825</td>
<td>62%</td>
</tr>
<tr>
<td>2</td>
<td>560</td>
<td>63%</td>
</tr>
<tr>
<td>3</td>
<td>325</td>
<td>37%</td>
</tr>
<tr>
<td>4</td>
<td>417</td>
<td>33%</td>
</tr>
<tr>
<td>5</td>
<td>191</td>
<td>28%</td>
</tr>
<tr>
<td>6</td>
<td>272</td>
<td>6%</td>
</tr>
</tbody>
</table>

Conditional distribution of $X$ given $Y=y$ is fair.

The research paper by Bickel, et al. concluded that women tended to apply to competitive departments with low rates of admission even among qualified applicants (such as in the English Department), whereas men tended to apply to less-competitive departments with high rates of admission among the qualified applicants (such as in engineering and chemistry). The conditions under which the admissions' frequency data from specific departments constitute a proper defense against charges of discrimination are formulated in the book *Causality* by Pearl.

- Inferring the marginal from conditional distr should be done carefully:
  \[
  f_X(x) = \int f(x,y) \, dy = \int f(x|y) \, f(y) \, dy
  \]
  depends on distr. of $Y$!

- Hence the following situation is possible:
  \[
  f_{X|Y}(x|y) \quad \text{trend reverses:}
  \]