6.7. Transformations of joint distributions

- Recall transformations of r.v's (univariate case — p.53)

  If $Y = g(X)$, and g is monotone, then

  $$f_Y(y) = f_X(x) \cdot \left| \frac{dg}{dx} \right|^{-1}, \quad y = g(x)$$

- Recall Proof:

$$P\{X \in A\} = \int_A f_X(x)\,dx = \left\{ \begin{array}{l} \text{change of var (g increases)} \\ y = g(x) \\ dy = \left(\frac{dg}{dx}\right) dx \Rightarrow dx = \left(\frac{dg}{dx}\right)^{-1} \end{array} \right\}$$

$$\Rightarrow P\{Y \in g(A)\} = \int_{g(A)} \underbrace{f_X(x) \cdot \left(\frac{dg}{dx}\right)^{-1}} dy \qquad (y = g(x))$$

$$\Rightarrow f_Y(y) = f_X(x) \cdot \left(\frac{dg}{dx}\right)^{-1}.$$

- Similarly if $Y_1 = g_1(X_1, X_2)$, $Y_2 = g(X_1, X_2)$, except the change of var is given by

  <u>Jacobian</u>
  $$J(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

  replacing $\left|\frac{dg}{dx}\right|$.

THM (Bivariate transformations of r.v's)

> Let X_1, X_2 be jointly continuous r.v.'s,
>
> let $Y_1 = g_1(X_1, X_2)$, $Y_2 = g(X_1, X_2)$
>
> for some continuously differentiable functions g_1, g_2 such that one can uniquely solve the system of equations
>
> $$y_1 = g_1(x_1, x_2), \quad y_2 = g_2(x_1, x_2) \qquad \text{for } x_1, x_2$$
>
> Then
>
> $$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}$$
>
> where $J(x_1, x_2) \neq 0$.

Ex [Ghahramani p. 357]

$X, Y \sim Exp(\lambda)$ independent.

$U = X+Y, \quad V = X/Y \quad \sim ?$

$f_X(x) = \lambda e^{-\lambda x}, \quad x>0$

$f_Y(y) = \lambda e^{-\lambda y}, \quad y>0.$

$f(x,y) = \lambda^2 e^{-\lambda(x+y)}, \quad x,y \geq 0.$

$U = g_1(X,Y), \quad V = g_2(X,Y)$ where $u = g_1(x,y) = x+y$, $v = g_2(x,y) = x/y$.

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 1/y & -\frac{x}{y^2} \end{vmatrix} = -\frac{x}{y^2} - \frac{1}{y} = -\frac{1}{y}\left(\frac{x}{y}+1\right) = -\frac{x+y}{y^2}$$

$$f_{U,V}(u,v) = \lambda^2 e^{-\lambda(x+y)} \frac{y^2}{x+y}$$

$$= \lambda^2 e^{-\lambda u} \frac{u}{(v+1)^2}, \quad u,v>0$$

Express $x+y=u$;

$v+1 = \frac{x}{y}+1 = \frac{x+y}{y} = \frac{u}{y} \Rightarrow$

$y = \frac{u}{v+1} \Rightarrow$

$\frac{y^2}{x+y} = \frac{u^2}{(v+1)^2 u} = \frac{u}{(v+1)^2}$

$x>0, y>0 \Leftrightarrow u,v>0$

$$= \left(\lambda e^{-\lambda u} \cdot \lambda u\right) \cdot \frac{1}{(v+1)^2}, \quad u,v>0$$

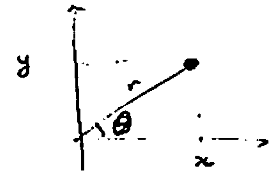
$$\boxed{U \sim Gamma(2, \lambda); \quad f_V(v) = \frac{1}{(v+1)^2}, \quad v>0 \quad \text{independent}.}$$

We knew this before as $U = X+Y$, $X,Y \sim Exp(\lambda)$.

Ex $X, Y \sim N(0,1)$ indep.

Polar coordinates : $x = r\cos\theta$, $y = r\sin\theta$

$R, \Theta \sim ?$



x, y by r, θ.

$$J(r,\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$\Rightarrow$ r, θ by x, y: reciprocal $\rightarrow$

$$J(x,y) = \frac{1}{r}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \quad \text{(bivariate Normal)}$$

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi} e^{-(x^2+y^2)/2} \cdot \left|\frac{1}{r}\right|^{-1} = \boxed{\frac{1}{2\pi} r e^{-r^2/2}, \quad r > 0,\ 0 < \theta < 2\pi}$$

$\Rightarrow$ $\Theta \sim \mathrm{Unif}(0, 2\pi)$, $\quad f_R(r) = re^{-r^2/2}, \quad r > 0$. $\leftarrow$ "Rayleigh" distr.

R, Θ independent!

Ex $R^2 \sim ?$

$Z = R^2$. $\quad z = r^2$

$$f_Z(z) = f_R(r) \left|\underbrace{\frac{dz}{dr}}_{= 2r}\right|^{-1} = re^{-r^2/2} \cdot \frac{1}{2r} = \frac{1}{2} e^{-r^2/2}$$

$$= \frac{1}{2} e^{-z/2} \sim \mathrm{Exp}\left(\frac{1}{2}\right)$$

$\Rightarrow$ Same result as in Ex. p.65 (Lec 03/11).

$$\boxed{\Theta \sim \mathrm{Unif}(0, 2\pi),\quad R^2 \sim \mathrm{Exp}\left(\tfrac{1}{2}\right) \text{ indep}}.$$

Ex $X_1, X_2 \sim N(0,1)$ indep

$Y_1 = X_1 + 2X_2, \quad Y_2 = 2X_1 - X_2$

Remark: By Σ Normal ind = Normal.
$Y_1, Y_2 \sim N(0, 1^2 + 2^2) = N(0,5)$.
But now want joint distn

$$g_1(x_1, x_2) = x_1 + 2x_2 = y_1$$
$$g_2(x_1, x_2) = 2x_1 - x_2 = y_2$$
$$J(x_1, x_2) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5.$$

$$\cancel{x_1 = \frac{2y_1 - y_2}{5}, \quad x_2 = \frac{y_1 + 2y_2}{5}} \qquad x_1 = \frac{y_1 + 2y_2}{5}, \quad x_2 = \frac{2y_1 - y_2}{5}.$$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{5} f_{X_1,X_2}\left(\frac{y_1 + 2y_2}{5}, \frac{2y_1 - y_2}{5}\right)$$

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2 + x_2^2)/2}$$

$$x_1^2 + x_2^2 = \frac{(y_1 + 2y_2)^2 + (2y_1 - y_2)^2}{25} = \frac{y_1^2 + 4y_1y_2 + 4y_2^2 + 4y_1^2 - 4y_1y_2 + y_2^2}{25}$$

$$= \frac{5y_1^2 + 5y_2^2}{25} = \frac{y_1^2 + y_2^2}{5}.$$

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{2\pi \cdot 5} e^{-(y_1^2 + y_2^2)/2\cdot 5} = \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{5}} e^{-y_1^2/2\cdot 5}}_{N(0,5)} \; \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{5}} e^{-y_2^2/2\cdot 5}}_{N(0,5)}$$

$\Rightarrow Y_1, Y_2 \sim N(0,5)$, $\boxed{\text{independent}}$

~~• Remark, we knew that $Y_1 \sim N(1$~~

• Discuss rotation invariance: transform. $U: (X_1, X_2) \mapsto \frac{1}{\sqrt{5}}(Y_1, Y_2)$, $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$
is unitary. $U(X) \sim N(0,1)$.