Properties of covariance

Prop $Cov(X,Y) = Cov(Y,X)$

$Cov(aX,Y) = a\, Cov(X,Y)$, $Cov(X,bY) = b\, Cov(X,Y)$

(This follows easily from def)

Prop (4.2) $Cov\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j Cov(X_i, Y_j)$ ← (Analog of the numeric identity $\left(\sum_i x_i\right)\left(\sum_j y_j\right) = \sum_i \sum_j x_i y_j$).

Proof $\mu_i := E[X_i]$, $\nu_j := E[Y_j]$

$$Cov\left(\sum_i X_i, \sum_j Y_j\right) = E\left[\left(\sum_i X_i\right)\left(\sum_j Y_j\right)\right] - E\left[\sum_i X_i\right] E\left[\sum_j Y_j\right] =: I - II$$

$$I = E\left[\sum_i \sum_j X_i Y_j\right] = \sum_i \sum_j E[X_i Y_j] \quad \text{(by linearity of expectation)}$$

$$II = \left(\sum_i E[X_i]\right)\left(\sum_j E[Y_j]\right) = \sum_i \sum_j E[X_i] E[Y_j]$$

$$\Rightarrow I - II = \sum_i \sum_j \left(E[X_i Y_j] - E[X_i] E[Y_j]\right) = \sum_i \sum_j Cov(X_i, Y_j).$$

QED.

Variance of a sum.

$Var\left(\sum_i X_i\right) \neq \sum_i Var(X_i)$ in general (for example, if $Y = -X$, $Var(X+Y) = 0$ but $Var(X) + Var(-X) > 0$)

Instead,

$$Var\left(\sum_i X_i\right) = Cov\left(\sum_i X_i, \sum_j X_j\right) = \sum_i \sum_j Cov(X_i, X_j) \quad \text{(by Prop. above)}$$

$$= \sum_i Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$$

We have proved:

Prop $Var\left(\sum_i X_i\right) = \sum_i Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$.

In particular, if X_i are uncorrelated then

$$Var\left(\sum_i X_i\right) = \sum_i Var(X_i)$$

Ex When one files a tax return, one has to add some n dollar amounts. If one rounds off each amount to the nearest whole dollar, what overall error should one expect?

Probabilistic model. D_i = exact amounts, (e.g. \$178.43)

Z_i = roundoff errors $\sim$ Unif $[-\frac{1}{2}, \frac{1}{2}]$ independent (−\$0.43)

- Each amount is rounded off to $X_i = D_i + Z_i$ (\$178)
- Total is $\sum_{i=1}^{n} X_i$

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) \quad \text{(by independence)}$$

$$= \sum_{i=1}^{n} \mathrm{Var}(Z_i) = \frac{n}{12} \quad \left(\text{since } \mathrm{Var}(Z_i) = \frac{1}{12}, \text{ see Uniform distribution}\right)$$

St dev of $\sum_{i=1}^{n} X_i$ is $\sqrt{\frac{n}{12}}$ = Ans

$\sqrt{\frac{n}{12}} \ll \frac{n}{2}$ (worst-case error) E.g. $n = 50$, $\sqrt{\frac{n}{12}} \approx \$2.0$ (while $\frac{n}{2} = \$25$)

- Application. quantization (probabilistic models)

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★ **Example 10.14 (Investment)** Dr. Caprio has invested money in three uncorrelated assets; 25% in the first one, 43% in the second one, and 32% in the third one. The means of the annual rate of returns for these assets, respectively, are 10%, 15%, and 13%. Their standard deviations are 8%, 12%, and 10%, respectively. Find the mean and standard deviation of the annual rate of return for Dr. Caprio's total investment.

Solution: Let r be the annual rate of return for Dr. Caprio's total investment. Let X_1, X_2, and X_3 be the annual rate of returns for the first, second, and third assets, respectively. By Example 4.25,

$$X = 0.25X_1 + 0.43X_2 + 0.32X_3.$$

Thus

$$\begin{aligned} E(X) &= 0.25E(X_1) + 0.43E(X_2) + 0.32E(X_3) \\ &= (0.25)(0.10) + (0.43)(0.15) + (0.32)(0.13) = 0.1311. \end{aligned}$$

Since the assets are uncorrelated, by (10.11),

$$\begin{aligned} \mathrm{Var}(X) &= (0.25)^2\,\mathrm{Var}(X_1) + (0.43)^2\,\mathrm{Var}(X_2) + (0.32)^2\,\mathrm{Var}(X_3) \\ &= (0.25)^2(0.08)^2 + (0.43)^2(0.12)^2 + (0.32)^2(0.10)^2 = 0.004087. \end{aligned}$$

Therefore, $\sigma_X = \sqrt{0.004087} = 0.064$. Hence Dr. Caprio should expect an annual rate of return of 13.11% with standard deviation 6.4%. Note that Dr. Caprio has reduced the standard deviation of his investments considerably by diversifying his investment; that is, by not putting all of his eggs in one basket.