7.7. Moment generating functions

Def. Mgf of a r.v. $X$ is the function

$$M(t) = M_X(t) := E[e^{tX}] , \quad t \in \mathbb{R}$$

Like cdf and pdf/pmf, the mgf contains all info about the
distribution of $X$. That is, mgf determines the distr. uniquely.

THM (Uniqueness) Let $X, Y$ be two r.v.'s. Assume that $\exists \delta > 0$ s.t

$$M_X(t) = M_Y(t) \text{ for all } t \in (-\delta, \delta)$$

Then $X, Y$ have the same distribution.

(wo proof - but mention analogy of $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx$ with Fourier transform,

Fourier inversion.)

How to get info about distr. of $X$ from mgf?

Differentiate:

$$M'(t) = \frac{d}{dt} E[e^{tX}] = E\left[\frac{d}{dt} e^{tX}\right] = E[X e^{tX}]$$

$$\Rightarrow \quad M'(0) = E[X]$$

$$\Rightarrow \quad M''(0) = \ldots = E[X^2] \Rightarrow \text{mgf determines mean, variance of } X$$

PROOF

$$M^{(n)}(0) = E[X^n] , \quad n = 1, 2, 3, \ldots$$

Remark: Local information

\[\rightarrow\]
(a) Bernoulli ($p$)

$$X = \begin{cases} 1, \text{ with prob. } p \\ 0, \text{ with prob. } 1-p \end{cases}$$

$$M(t) = E[e^{tX}] = e^{t}p + e^{-t}(1-p) = pe^t + 1 - p.$$

(e.g. for $p = \frac{1}{2}$, $M(t) = \frac{e^t + 1}{2}$)

$$M'(t) = pe^t \Rightarrow E(X) = M'(0) = p \quad \text{(as we know)}$$

$$M''(t) = pe^t \Rightarrow E(X^2) = p.$$  

$\Rightarrow \text{Var}(X) = p - p^2 = p(1-p)$

(b) Normal $\mathcal{N}(0,1)$

$$M(t) = E[e^{tx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2} \, dx = \ldots \text{ (complete the square)}$$

$$= e^{t^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} \, dx = e^{t^2/2}$$

$$M'(t) = e^{t^2/2} \cdot t \quad E[X] = M'(0) = 0$$

$$M''(t) = e^{t^2/2} \cdot t + e^{t^2/2} = e^{t^2/2} \left( t^2 + 1 \right) \quad E[X^2] = M''(0) = 1.$$  

$\Rightarrow \text{Var}(X) = 1 - 0 = 1$ (as we know)

(c) Normal $X \sim \mathcal{N}(\mu, \sigma^2)$ $\Rightarrow x \sim \mathcal{N}(x, 0 Z$)

$M_X(t) = E(e^{tX}) = e^{\mu t} E[e^{t^2 \sigma^2}] = e^{\mu t} M_Z(t)$

$= e^{\mu t} \cdot e^{t^2 \sigma^2}$

$= e^{\mu t + \mu^2 \sigma^2}$

Prop

$X_1, \ldots, X_n$ independent r.v.'s. Then

$$M_{X_1 + \ldots + X_n}(t) = M_{X_1}(t) \ldots M_{X_n}(t)$$

Prop

$$M_{X_1, \ldots, X_n}(t) = E[e^{t(X_1 + \ldots + X_n)}] = E[e^{tX_1} \ldots e^{tX_n}]$$

$$= E[e^{tX_1}] \ldots E[e^{tX_n}] \quad \text{(by independence)}$$

$$= M_{X_1}(t) \ldots M_{X_n}(t)$$
Ex

(a) $X \sim \text{Binom} (n, p)$ \implies X = X_1 + \ldots + X_n, \quad X_i \sim \text{Bernoulli} (p)$ iid.

$M_X (t) = \prod_{i=1}^{n} M_{X_i} (t) = (pe^t + 1 - p)^n$ \hspace{1cm} (by Ex.c, p.103)

$M_X' (t) = n (pe^t + 1 - p)^{n-1} pe^t$

$E[X] = M'(0) = (np)$ \hspace{1cm} (as we know)

(b) Let $X_1 \sim N (\mu_1, \sigma_1^2), \quad X_2 \sim N (\mu_2, \sigma_2^2)$.

$X := X_1 + X_2 \sim ?$

$M_{X_1+X_2} (t) = M_{X_1} (t) \cdot M_{X_2} (t) = \exp \left( \frac{\sigma_1^2 t^2}{2} + \mu_1 t \right) \cdot \exp \left( \frac{\sigma_2^2 t^2}{2} + \mu_2 t \right)$

$\hspace{3cm} = \exp \left[ \frac{(\sigma_1^2 + \sigma_2^2) t^2}{2} + (\mu_1 + \mu_2) t \right]$. \hspace{1cm} (by Ex.c, p.103)

This is the mgf of $N (\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Hence, by the uniqueness theorem,

$X \sim N (\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ \hspace{1cm} (as we know - but now with a simpler proof, which also carries over to more than 2 r.v.'s)

More examples of computing rgs (Poisson, geometric, exponential, etc.) — see the book & Wikipedia.