04/09 Lec 37

7.7. Moment generating functions

Def Mgf of a r.v X is the function

$$M(t) = M_X(t) := E[e^{tX}], \quad t \in \mathbb{R}$$

Like cdf and pdf/pmf, the mgf contains all info about the distribution of X. That is, mgf determines the distr. uniquely.

THM (Uniqueness) Let X, Y be two r.v's. Assume that $\exists\, \delta > 0$ s.t

$$M_X(t) = M_Y(t) \quad \text{for all } t \in (-\delta, \delta)$$

Then X, Y have the same distribution.

(w/o proof — but mention analogy of $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$ with Fourier transform, $\Rightarrow$ Fourier inversion)

- How to get info about distr of X from mgf?

Differentiate:

- $M'(t) = \frac{d}{dt} E[e^{tX}] = E[\frac{d}{dt} e^{tX}] = E[X e^{tX}]$.

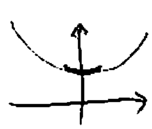
$\Rightarrow$ $$\boxed{M'(0) = E(X)}$$

- $M''(0) = \cdots = E[X^2]$ $\Rightarrow$ mgf determines mean, variance of X

PROP $M^{(n)}(0) = E(X^n)$, $n = 1, 2, 3, \ldots$

"n'th moment of X"

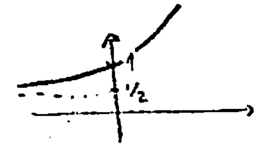
Remark local information



Ex (a) Bernoulli(p) $\quad X = \begin{cases} 1, & \text{with prob. } p \\ 0, & \text{with prob. } 1-p. \end{cases}$

$$M(t) = E[e^{tX}] = e^{t\cdot 1}\cdot p + e^{t\cdot 0}\cdot(1-p) = pe^t + 1 - p.$$

$$\left(\text{e.g. for } p = \tfrac{1}{2},\quad M(t) = \frac{e^t+1}{2}\right)$$



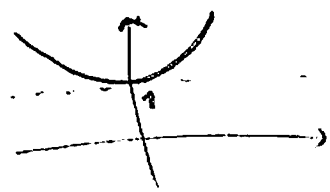
$M'(t) = pe^t \Rightarrow E[X] = M'(0) = pe^0 = p$.

$M''(t) = pe^t \Rightarrow E[X^2] = p. \Rightarrow Var(X) = p - p^2 = p(1-p)$ (as we know)

(b) Normal $Z \sim N(0,1)$:

$$M(t) = E[e^{tZ}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\, dx = \dots \text{(complete the square)}$$

$$= e^{t^2/2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2}\, dx}_{1} = e^{t^2/2}$$



$M'(t) = e^{t^2/2}\cdot t \qquad E[X] = M'(0) = 0$

$M''(t) = e^{t^2/2}\, t\cdot t + e^{t^2/2} = e^{t^2/2}(t^2+1). \qquad E[X^2] = M''(0) = 1.$

$\Rightarrow Var(Z) = 1 - 0 = 1$ (as we know)

(c) Normal $X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma Z \qquad M_X(t) = E[e^{t(\mu+\sigma Z)}] = e^{t\mu} E[e^{t\sigma Z}] = e^{t\mu} M_Z(t\sigma)$

$= e^{t\mu}\cdot e^{t^2\sigma^2/2}$

$= \boxed{e^{\frac{\sigma^2 t^2}{2} + \mu t}}$

Prop $X_1, \dots, X_n$ indep. r.v.'s. Then

$$M_{X_1+\dots+X_n}(t) = M_{X_1}(t)\cdots M_{X_n}(t)$$

Proof $M_{X_1+\dots+X_n}(t) = E[e^{t(X_1+\dots+X_n)}] = E[e^{tX_1}\cdots e^{tX_n}]$

$= E[e^{tX_1}]\cdots E[e^{tX_n}]$ (by indep.)

$= M_{X_1}(t)\cdots M_{X_n}(t)$

Ex (a) $X \sim \text{Binom}(n,p) \Rightarrow X = X_1 + \dots + X_n$, $X_i \sim \text{Bernoulli}(p)$ iid.

$$M_X(t) = M_{X_1}(t) \cdots M_{X_n}(t) = \boxed{(pe^t + 1 - p)^n}.$$ (by Ex. a p.103)

$$M_X'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$E[X] = M'(0) = np.$$ (as we know)

(b) Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$.

$$X := X_1 + X_2 \sim ?$$

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = \exp\left(\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right) \times \exp\left(\frac{\sigma_2^2 t^2}{2} + \mu_2 t\right)$$ (by Ex. c p.103)

$$= \exp\left[\frac{(\sigma_1^2 + \sigma_2^2) t^2}{2} + (\mu_1 + \mu_2) t\right].$$

This is the mgf of $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ (by Ex. c, p.103).

Hence, by the uniqueness theorem,

$$X \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_1^2)$$

(as we know – but now with a simpler proof, which also carries over to <u>more than</u> 2 r.v.'s)

<u>More examples</u> of computing mgf's (Poisson, geometric, exponential, etc.) — see the book & Wikipedia.