§10. Monotone sequences

Def (s_n) is *nondecreasing* if $s_n \le s_{n+1}$ for all n, *increasing* if "$>$"
nonincreasing if $s_n \ge s_{n+1}$ for all n. decr. if "$<$".
Monotone

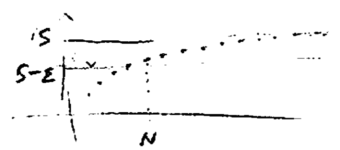
Remark: nondecr: $s_n \le s_m$ for all $n < m$ (by induction)

Examples (a) $s_n = \frac{1}{n^2}$ $\downarrow$

(b) $s_n = \sqrt{1+n^2}$ $\uparrow$

(c) $s_n = 10n - e^n$ —

Thm 10.2 $\boxed{\text{A bounded monotone sequence converges}}$



WLOG: (s_n) $\uparrow$. $S := \{s_n : n \in \mathbb{N}\}$.

Will prove: $\lim s_n = \sup S =: s$ $(\exists)$.

Let $\varepsilon > 0$. $s - \varepsilon$ is not an upper bd for S $\Rightarrow$

$\exists N$ $s_N > s - \varepsilon$.

Non-decreasing $\Rightarrow$ $s_n > s_N > s - \varepsilon$ for $n > N$.

Also, $s_n \le s < s + \varepsilon$ (s is an upper bd)

$\Rightarrow$ $|s_n - s| < \varepsilon$ for $n > N$. QED

Remark $s_n \uparrow \Rightarrow$ ~~converges to~~ $\lim s_n = \sup\{s_n\}$.
$s_n \downarrow \Rightarrow \lim s_n = \inf\{s_n\}$.

Discussion of decimals $\pi = 3.1415926\ldots$

Every ∞ expression $s = k.d_1 d_2 d_3 d_4 \ldots$ represents a real number. **Why?**
$d_j \in \{0, 1, \ldots, 9\}$

Truncate at n, $s_n := k + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n}$.

(a) (s_n) non-decreasing

(b) (s_n) bounded: $s_n < k + \underbrace{\frac{9}{10} + \frac{9}{10^2} + \ldots \frac{9}{10^n}}_{\le 1} < k+1$ (by geom series).

$\Rightarrow$ $\lim s_n$ exists by Thm 10.2, called $\boxed{s}$.

Exercise (Thm 10.4 a) : (a) (s_n) unbounded, non-decreasing $\Rightarrow \lim s_n = +\infty$

(b) (s_n) unbdd, non-increasing $\Rightarrow \lim s_n = -\infty$.

e

Example $\lim \left(1+\frac{1}{n}\right)^n$ exists. Called e (for Euler), $e \approx 2.718281828459045\ldots$

Proof of convergence:

1) $s_n = \left(1+\frac{1}{n}\right)^{n+1}$ decreasing? $(s_n) = (4, 3.16\ldots, 3.05\ldots, 2.99\ldots, 2.94\ldots)$

$$s_{n-1} = \left(1+\frac{1}{n-1}\right)^{n} > \left(1+\frac{1}{n}\right)^{n+1} = s_n \quad ?$$

$$\left(\frac{1+\frac{1}{n-1}}{1+\frac{1}{n}}\right)^n > 1+\frac{1}{n} \quad ?$$

$$\| $$

$$\left[\frac{n\cdot n}{(n-1)(n+1)}\right]^n = \left[\frac{n^2}{n^2-1}\right]^n = \left(1+\frac{1}{n^2-1}\right)^n \geq 1 + n\cdot\frac{1}{n^2-1} \quad \left(\text{by Bernoulli ineq } (1+a)^n \geq 1+an\right)$$

$$\geq 1+\frac{1}{n} \ ?$$

$$\frac{n}{n^2-1} > \frac{1}{n} \quad \Rightarrow \text{Yes.}$$

2) $s_n \geq 0 \Rightarrow$ Bdd below

$\Rightarrow \lim s_n$ exists by Weierstrass Theorem.

Finally,
$$\lim \left(1+\frac{1}{n}\right)^n = \lim \frac{s_n}{1+1/n} = \frac{\lim s_n}{\lim\left(1+\frac{1}{n}\right)} = \lim s_n. \qquad \text{Q.E.D.}$$