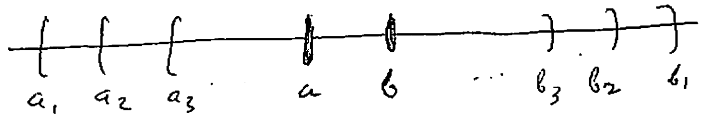
10/7/2011

# Nested Intervals Theorem.

**Def** Subsets $I_1, I_2, \ldots$ in $\mathbb{R}$ are <u>nested</u> if $I_1 \supset I_2 \supset \ldots$

**Thm** (<u>Nested Intervals Thm</u>). Let $I_1, I_2, \ldots$ be nested <u>non-empty bounded closed intervals in $\mathbb{R}$</u>".
(i.e. $I_n = [a_n, b_n]$, $a_n < b_n$.)

> Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty,
> i.e. there exists $x \in \mathbb{R}$ that belongs to <u>all</u> $I_n$.
> Moreover, if the length $|I_n| \to 0$ then ~~there i~~ such $x$ is unique.



<u>Proof</u> $I_n = [a_n, b_n]$ nested $\Rightarrow$ $(a_n)$ non-decreasing, $(b_n)$ non-increasing.

Weierstrass Theorem (10.2) implies the existence of

$$\lim a_n =: a, \quad \lim b_n =: b.$$

Moreover, ~~Recall~~ by Remark to the Weierstrass Theorem (p. 27),

$$a = \sup\{a_n : n \in \mathbb{N}\}, \quad b = \inf\{b_n : n \in \mathbb{N}\}$$

Since $a_n \le b_n$ for all $n$, we have also that $a \le b$ (see Exercise 4.8).

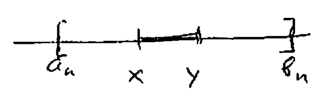
Thus

$$a_n \le a \le b \le b_n \quad \text{for all } n.$$

So $[a, b] \subseteq [a_n, b_n] = I_n$ for all $n$.

Hence $[a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$ This proves the first part of N.I.T

<u>"Moreover" part</u>: Let $x, y \in \bigcap_{n=1}^{\infty} I_n$



Then $|x-y| \le |I_n| = b_n - a_n$ for all $n$.

Since $\lim(b_n - a_n) = 0$, the Squeeze Theorem implies that

$$\lim |x-y| = 0. \quad \text{But} \quad \lim |x-y| = |x-y|.$$

Hence $|x-y| = 0$, so $x = y$. Q.E.D.

(optimality)
**Remarks** (a) N.I.T. might not hold if "closed" is omitted.

Example: $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$.

(b) N.I.T. might not hold if "bounded" is omitted.

Example: $\bigcap_{n=1}^{\infty} (n, \infty) = \emptyset$

(c) N.I.T. might not hold ~~of over a all~~ over $\mathbb{Q}$

(i.e. if all numbers involved were rational rather than real):

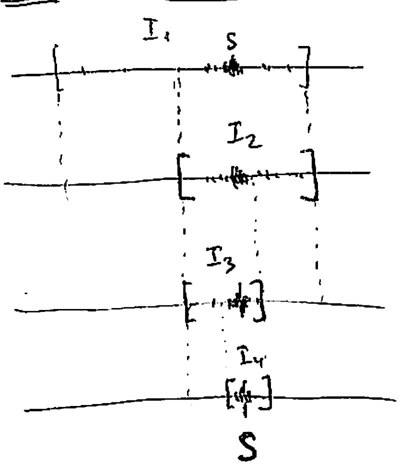
Example: $\bigcap_{n=1}^{\infty} \{x \in (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}),\ x \in \mathbb{Q}\} = \sqrt{2} \notin \mathbb{Q}$.

**Thm (1.15) (Bolzano-Weierstrass)**

Every bounded sequence $(s_n)_{n=1}^{\infty}$ of real numbers has a convergent subsequence $(s_{n_k})_{k=1}^{\infty}$.

For a detailed discussion of subsequences, see §11.

**Proof** "Divide and conquer" strategy:




- Since $(s_n)$ is bounded, there exists a closed bounded interval $I_1$ s.t. $\{s_n : n \in \mathbb{R}\} \subseteq I_1$. $(*)$
- Divide $I_1$ into two halves. One of them (call it $I_2$) must contain infinitely many terms of the sequence $(s_n)$. ~~i.e. call it $I_2$~~ i.e. $|\{n : s_n \in I_2\}| = \infty$ $(**)$
- Divide $I_2$ into two halves. One of them (call it $I_3$) must contain $\infty$ many terms of $(s_n)$

etc. . . . . .

- By N.I.T, $\bigcap_{n=1}^{\infty} I_n$ is non-empty; so $\exists\, S \in \mathbb{R}$ s.t. $\boxed{S \in \bigcap_{n=1}^{\infty} I_n}$.

Construction of $(s_{n_k})$: ~~Choose~~ $n_1 := 1$; $s_{n_1} \in I_1$ by $(*)$.

~~Choose~~ By $(**)$, there exists $n_2 > n_1$ s.t. $s_{n_2} \in I_2$

Similarly, there exists $n_3 > n_2$ s.t. $s_{n_3} \in I_3$

etc. . . . . . $n_k > n_{k-1}$ s.t. $s_{n_k} \in I_{n_k}$.

Since $s_{n_k}, S \in I_k$, we have

$$0 \le |s_{n_k} - S| \le |I_k| \to 0 \text{ as } k \to \infty.$$

By Squeeze Thm, $|s_{n_k} - S| \to 0$ as $k \to \infty$ $\Rightarrow$ $\lim (s_{n_k} - S) = 0$ $\Rightarrow$ $\lim s_n = S$ Q.E.D.