Nested Intervals Theorem.

Def. Let $I_1, I_2, \ldots$ be nested non-empty bounded closed intervals in $\mathbb{R}$.

Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty, i.e., there exists $x \in \mathbb{R}$ that belongs to all $I_n$.

Moreover, if the length $|I_n| \to 0$ then there is such $x$ is unique.

\[ a_1, a_2, a_3, a, b, b_2, b_3, \ldots \]

Proof: $I_n = [a_n, b_n]$ nested $\implies (a_n)$ non-decreasing, $(b_n)$ non-increasing.

By Weierstrass Theorem (10.2) implies $\exists$ existence of

\[ \lim a_n = a, \quad \lim b_n = b. \]

Moreover, by Remark to the Weierstrass Theorem (p. 27),

\[ a = \sup \{a_n : n \in \mathbb{N}\}, \quad b = \inf \{b_n : n \in \mathbb{N}\}. \]

Since $a_n \leq b_n$ for all $n$, we have also that $a \leq b$ (see Exercise 4.8).

Thus

\[ a_n \leq a \leq b \leq b_n \]

for all $n$.

So

\[ [a, b] \subseteq [a_n, b_n] = I_n \]

for all $n$.

Hence

\[ [a, b] \subseteq \bigcap_{n=1}^{\infty} I_n. \]

This proves the first part of NIT.

Moreover, part: Let $x, y \in \bigcap_{n=1}^{\infty} I_n$\[ a_n \]

Then

\[ |x-y| \leq |I_n| = b_n - a_n \]

for all $n$.

Since $\lim(b_n - a_n) = 0$, the Squeeze Theorem implies that

\[ \lim |x-y| = 0. \]

But $\lim |x-y| = |x-y|$, hence

\[ \lim |x-y| = 0, \quad \text{so } x = y. \]

\[ \text{Q.E.D.} \]
Remarks: (a) N.I.T. might not hold if "closed" is omitted.

Example: \( \cap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset \).

(b) N.I.T. might not hold if "bounded" is omitted.

Example: \( \cap_{n=1}^{\infty} (n, \infty) = \emptyset \).

(c) N.I.T. might not hold if \( \mathbb{Q} \) is replaced by \( \mathbb{Q} \) (i.e., if all numbers involved were rational rather than real).

Example: \( \cap_{n=1}^{\infty} (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}), x \in \mathbb{Q} \), so \( \sqrt{2} \notin \mathbb{Q} \).

Thus (11.5) (Bolzano–Weierstrass):

Every bounded sequence \( (s_n)_{n=1}^{\infty} \) of real numbers has a convergent subsequence \( (s_{n_k})_{k=1}^{\infty} \).

For a detailed discussion of subsequences, see §11.

Proof. "Divide and conquer" strategy:

\[
\begin{array}{c|c|c|c}
 & I_1 & S & I_2 \\
\hline
| & | & | & | \\
\hline
& I_3 & S & I_4 \\
\hline
& & & &
\end{array}
\]

- Since \( (s_n) \) is bounded, there exists a closed bounded interval \( I_1 \) s.t.
  \( \{s_n: n \in \mathbb{N}, s_n \in I_1\} \neq \emptyset \).

- Divide \( I_1 \) into two halves, one of which must contain infinitely many terms of the sequence \( (s_n) \), i.e.,
  \( \exists n_1 \in \mathbb{N}, |I_1| = \infty \).

- Divide \( I_2 \) into two halves, one of which (call it \( I_2 \)) must contain \( \infty \) many terms of \( (s_n) \).

Construction of \( (s_{n_k}) \):

Choose: \( n_1 = 1 \), \( S_n \in I_1 \) by (11.5).

Choose by (11.5), there exists \( n_2 > n_1 \) s.t. \( s_{n_2} \in I_2 \).

Similarly, there exists \( n_3 > n_2 \) s.t. \( s_{n_3} \in I_3 \).

\( \vdots \ldots \).

Since \( s_{n_k} \in I_k \), we have
\[ 0 \leq |s_{n_k} - s| \leq |I_k| \rightarrow 0 \text{ as } k \rightarrow \infty. \]

By Squeeze Theorem, \( s_{n_k} \rightarrow s \) as \( k \rightarrow \infty \).

\[ \lim_{k \rightarrow \infty} (s_{n_k} - s) = 0, \quad \lim_{k \rightarrow \infty} s_{n_k} = s \quad \square. \]