10/10/2011

Cauchy Sequences (§10)

- Recall Bolzano-Weierstrass: every bounded sequence has a convergent subsequence.
  May have many convergent subsequences, e.g. $(-1)^n$ has two. Some have ∞ many!
- Inconvenient. Caused by oscillations
- Need a condition that prevents oscillations.

  Cauchy: terms are close to each other.

Def 10.8. (s_n) is a Cauchy sequence if:
For every $\varepsilon > 0$ there exists N such that
$$|s_n - s_m| < \varepsilon \quad \text{for } n,m > N$$

- Compare to def of limit: no reference to the limit point s! No need to know s.

Thm 10.11 A sequence (s_n) converges if and only if (s_n) is Cauchy.

Proof of $(\Rightarrow)$. Assume $\lim s_n = s$, let's verify the def. of Cauchy.
Let $\varepsilon > 0$. Choose N by def. of limit so that
$$|s_n - s| < \frac{\varepsilon}{2} \quad \text{for } n > N.$$
Then, for every $n, m > N$,
$$|s_n - s_m| < |s_n - s| + |s - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{QED.}$$

The proof of $(\Leftarrow)$ will be based on Bolzano-Weierstrass thm.
We need two lemmas.

<u>Lemma 10.10</u> Cauchy sequences are Bounded

<u>Proof</u> Use def. of Cauchy with $\varepsilon = 1$: there exists N such that

$$|S_n - S_m| < 1 \quad \text{for } n, m > N.$$

Thus $\quad |S_n - S_{N+1}| \quad$ for $n > N$

so

$$|S_n| < |S_{N+1}| + 1 \quad \text{for } n > N.$$

Hence $\{S_n : n > N\}$ is Bounded. Since there are finitely many terms in $\{S_n : n < N\}$, the whole sequence (S_n) is Bounded.

<u>Proof of (Thm 10.11 $\Leftarrow$)</u>: Assume (S_n) is Cauchy.

By Lemma 10.10, (S_n) is Bounded.

- By Bolzano-Weierstrass theorem, (S_n) has a convergent subsequence (S_{n_k}); say $\lim_{k \to \infty} S_{n_k} = S$. $\quad (*)$

- It remains to show that $\lim_{n \to \infty} S_n = S$.

  Let $\varepsilon > 0$ be arbitrary.

  - By $(*)$, there exists K such that

    $$|S_{n_k} - S| < \varepsilon/2 \quad \text{for } k > K. \qquad (**)$$

  - Since (S_n) is Cauchy, there exists N such that

    $$|S_n - S_m| < \varepsilon/2 \quad \text{for } n, m > N.$$

Then,

$$|S_n - S| \leq |S_n - S_{n_k}| + |S_{n_k} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n > N$.

(Here we chose k s.t. $k > K$, $n_k > N$.)

QED.

Exercise Let (s_n) be a sequence such that, for some $a<1$, one has

$$|s_n - s_{n+1}| \le a^n \text{ for all } n.$$

Prove that (s_n) converges.

Hint: ~~Write $s_n = s_n$ Show that (s_n)~~

It suffices to show that (s_n) is Cauchy.

To show this, write

$$|s_n - s_m| = |(s_n - s_{n+1}) + (s_{n+1} - s_{n+2}) + \dots + (s_{m-1} - s_m)|$$

and use triangle inequality and summation of geometric progression.

Remark A sequence that satisfies

$$\lim (s_n - s_{n+1}) = 0$$

is not necessarily Cauchy

(Example: $s_n = \log n$; then $s_n - s_{n-1} = \log \frac{n+1}{n} = \log\left(1 + \frac{1}{n}\right) = 0$).
but (s_n) is unbounded

10/10/2011

Application: Fixed point Theorem

The following is an application of the theory of limit to analytic problems. The result below is due to Stephan Banach, and it holds for R^n and more general spaces ("Banach spaces"). We state its partial case for R^1.

Theorem (Fixed point theorem) Suppose that f is a function on R such that

$$|f(x)-f(y)| \le a|x-y| \quad \text{for all } x, y. \qquad (\text{"Contraction"})$$

Then f has a fixed point, i.e. there exists $x \in R$ such that

$$f(x) = x.$$

Proof • Choose $x_0 \in R$ arbitrary, and let

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \dots \qquad \text{"Orbit of } x_0\text{"}$$

• $|x_2 - x_1| = |f(x_1) - f(x_0)| \le a|x_1 - x_0|$.

$|x_3 - x_2| = |f(x_2) - f(x_1)| \le a|x_2 - x_1| \le a^2|x_1 - x_0|$

$\dots\dots$

$|x_{n+1} - x_n| \le a^n |x_1 - x_0|$ for all n. (Exercise: prove this rigorously by induction.)

Hence (x_n) is Cauchy, so (x_n) converges.

• Let $\boxed{\lim x_n = x}$

Since $(f(x_n))_{n=1}^{\infty} = (x_{n+1})_{n=1}^{\infty}$, we also have

$$\boxed{\lim f(x_n) = x}. \qquad (*)$$

• $|f(x_n) - f(x)| \le a|x_n - x| \to 0$ as $n \to \infty$

So by Squeeze Theorem, $\lim f(x_n) = f(x)$. By $(*)$ and the uniqueness of limit, $f(x) = x$.

Q.E.D.