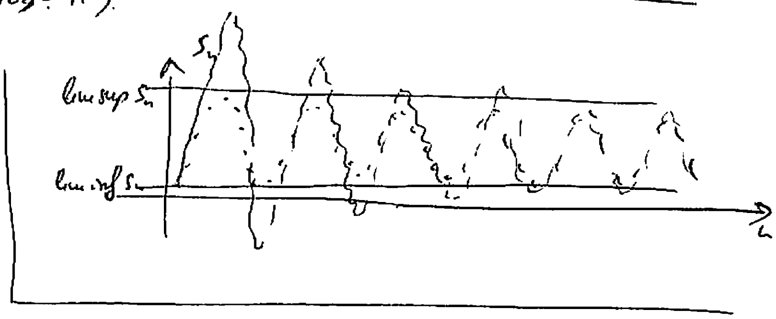
10/12/2011

Lim sup and Lim inf (§10, §11)

- Even if a sequence diverges/oscillates, it is possible to define a "proxy" for the notion of limit.



Actually, two proxies:

lim inf s_n and lim sup s_n.

Def 10.6. Let (s_n) be a sequence.

- The limit supremum of (s_n) is defined as

$$\limsup s_n = \lim_{n\to\infty}\Big(\underbrace{\sup_{k\geq n} s_k}\Big) \longrightarrow \sup\{x_k : k\geq n\}.$$

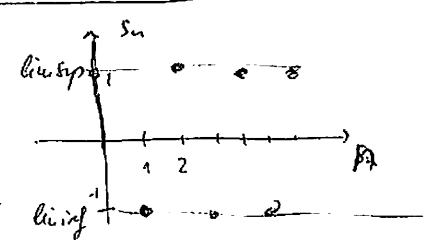
- The limit infimum of (s_n) is defined as

$$\liminf s_n = \lim_{n\to\infty}\Big(\inf_{k\geq n} s_k\Big).$$

Examples: (a) $s_n = (-1)^n$

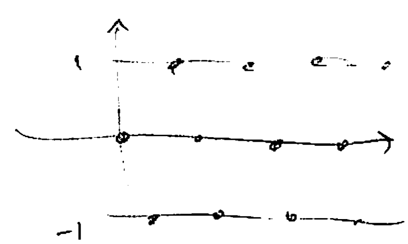
$$\limsup s_n = \lim_{n\to\infty}\Big(\underbrace{\sup_{k\geq n} (-1)^n}_{=1}\Big) = 1.$$

$$\liminf s_n = \lim_{n\to\infty}\Big(\underbrace{\inf_{k\geq n} (-1)^n}_{=-1}\Big) = -1$$

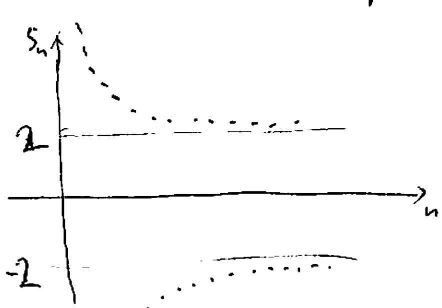


(b) $s_n = \cos\left(\frac{n\pi}{2}\right)$; $(s_n) = (0, -1, 0, 1, 0, -1, 0, 1, \dots)$

lim sup $(s_n) = 1$, lim inf $(s_n) = -1$.



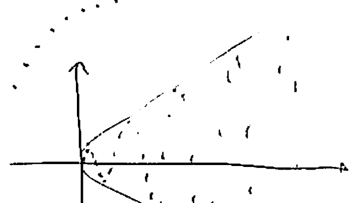
(c)



$$s_n = (-1)^n\left(\frac{n+2}{n}\right)$$

lim sup $s_n = 2$, lim inf $s_n = -2$.

(d)



$$s_n = \cos\left(\frac{n\pi}{10}\right)\cdot n$$

lim sup $s_n = \infty$, lim $s_n = -\infty$.

Prop. For every sequence (s_n), $\limsup s_n$ and $\liminf s_n$ exist — either as real numbers (if s_n is bounded), or $\pm\infty$ (if s_n is unbounded).

Proof By def., $\limsup s_n = \lim_{n\to\infty} S_n$ where $S_n = \sup_{k>n} s_k$.

(S_n) is a non-increasing sequence by definition; [Check!] it is bounded if and only if (s_n) is bounded.

By Weierstrass theorem, S_n converges $\Rightarrow$ $\limsup s_n$ exists.

An argument for $\liminf s_n$ is similar, as $T_n := \inf_{k>n} s_k$ is non-decreasing [Check!]

QED

Remark $\limsup s_n \le \sup s_n$, $\liminf s_n \ge \inf s_n$

BUT need not be equal (as in Example C on p.36).

$\limsup s_n$ is the value that ∞ many of s_n ~~need to~~ get close to.

~~sup s_n~~ ~~(as opposed of~~

Theorem (≤11.7) Let $s = \limsup s_n$, $t = \liminf s_n$

There exists a subsequence $s_{n_k} \to s$ as $k \to \infty$,

There exists a subsequence $s_{l_j} \to s$ as $j \to \infty$.

Check for the examples on p. 36.

Proof (for lim sup and for finite s, t; exercise: prove for $s, t = \pm\infty$ and for liminf).

~~Proof for sup~~

By def,
$$s = \lim S_n \quad \text{where} \quad S_n = \sup_{k>n} s_k.$$

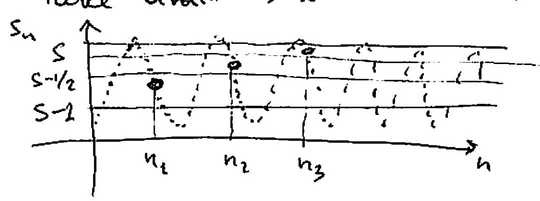
Since $S_0 = \sup_{k>0} s_k$, there exists (by def. of sup) $n_1 \in \mathbb{N}$ such that $\boxed{S_0 - 1 \le s_{n_1} \le S_0}$

Similarly, since $S_{n_1} = \sup_{k>n_1} s_k$, there exists $n_2 > n_1$ such that $\boxed{S_{n_1} - \frac{1}{2} \le s_{n_2} \le S_{n_1}}$

. . . . . .

there exist $n_k > n_{k-1}$ such that $\boxed{S_{n_{k-1}} - \frac{1}{2^{k-1}} \le s_{n_k} \le S_{n_{k-1}}}$

Take limit as $k \to \infty$ and use Squeeze Thm.; $\begin{cases} S_{n_k} \to s \text{ by def;} \\ S_{n_k} - \frac{1}{2^{k-1}} \to s \end{cases}$ hence $s_{n_k} \to s$ as well.



QED

Thm 10.7 For a sequence (s_n),

(s_n) converges and $\lim s_n = s$
if and only if $\liminf s_n = \limsup s_n = s$.

Proof ($\Rightarrow$). Suppose $\lim s_n = s$, $\limsup s_n = s'$. W.T.S: $s = s'$.

By Theorem ($\leq$11.7 p.37), there exists a subsequence

$$s_{n_k} \to s' \quad \text{as } k \to \infty.$$

But since $\lim s_n = s$, $s_{n_k} \to s$ as well. Hence $s = s'$.

The argument for $\liminf s_n$ is similar. (give it!)

($\Leftarrow$). Assume $\liminf s_n = \limsup s_n = s$. WTS: $s_n \to s$. We check that (s_n) is Cauchy.

Let $\varepsilon > 0$; ~~choose~~ by def of liminf, limsup we can choose N such that

$$\left|\sup_{k>n} s_k - s\right| < \frac{\varepsilon}{2}, \quad \left|\inf_{k>m} s_k - s\right| < \frac{\varepsilon}{2} \quad \text{for } n, m > N \qquad (*)$$

Let $n, m > N$, WLOG $s_n > s_m$.

$$|s_n - s_m| = s_n - s_m < \left(s + \frac{\varepsilon}{2}\right) - \left(s - \frac{\varepsilon}{2}\right) \qquad (\text{using } (*))$$

$$= \varepsilon.$$

Thus (s_n) is Cauchy, hence converges.

Moreover, by Thm ($\leq$11.7 p.37), some sub-sequence of (s_n) converges to s. Hence (s_n) converges to s. QED.

Thm ($\sim$11.7) $\limsup s_n$ is the largest value to which some subsequence of (s_n) converges.
$\liminf s_n$ is the smallest value to which some subsequence of (s_n) converges.
Precisely, if $s_{n_k} \to s$ as $k \to \infty$ then

$$\liminf s_n \leq s \leq \limsup s_n$$

Proof Let $s_{n_k} \to s$ as $k \to \infty$. Fix N and choose k: $n_k > N$. Then

$$\inf_{j>N} s_j \leq s_{n_k} \leq \sup_{j>N} s_j$$

Take limits ~~of all sides~~ as $k \to \infty$; by a comparison theorem,

$$\inf_{j>N} s_j \leq s \leq \sup_{j>N} s_j.$$

Now take limit of both sides as $N \to \infty$; we obtain $\liminf s_n \leq s \leq \limsup s_n$. QED