Even if a sequence diverges/oscillates, it is possible to define a "proxy" for the notion of limit.

Actually, two proxies: \(\lim \inf s_n\) and \(\lim \sup s_n\).

**Def 10.6.** Let \((s_n)\) be a sequence.

- The **limit supremum** of \((s_n)\) is defined as

\[
\limsup s_n = \lim_{n \to \infty} \left( \sup_{k \geq n} s_k \right)
\]

- The **limit infimum** of \((s_n)\) is defined as

\[
\liminf s_n = \lim_{n \to \infty} \left( \inf_{k \geq n} s_k \right)
\]

**Examples:**

(a) \(s_n = (-1)^n\)

\[
\limsup s_n = \lim_{n \to \infty} \left( \sup_{k \geq n} (-1)^n \right) = 1,
\]

\[
\liminf s_n = \lim_{n \to \infty} \left( \inf_{k \geq n} (-1)^n \right) = -1.
\]

(b) \(s_n = \cos \left( \frac{n \pi}{2} \right)\):

\[
(s_n) = (0, -1, 0, 1, 0, -1, 0, 1, \ldots)
\]

\[
\limsup s_n = 1,
\]

\[
\liminf s_n = -1.
\]

(c) \(s_n = (-1)^n \left( \frac{n-1}{n} \right)\)

\[
\limsup s_n = 2,
\]

\[
\liminf s_n = -2.
\]

(d) \(s_n = \cos \left( \frac{n \pi}{10} \right)\)

\[
\limsup s_n = \infty,
\]

\[
\liminf s_n = -\infty.
\]
Prop: For every sequence \((s_n)\), \(\inf s_n\) and \(\sup s_n\) exist — either as real numbers (if \((s_n)\) is bounded) or \(\pm \infty\) (if \((s_n)\) is unbounded).

Proof: By def., \(\sup s_n = \lim_{n \to \infty} s_n\) where \(s_n = \sup_{K>n} s_K\).

\((s_n)\) is a non-increasing sequence by def.; it is bounded if and only if \((s_n)\) is bounded.

By Weierstrass theorem, \(s_n\) converges ⇒ \(\inf s_n\) exists.

An argument for \(\inf s_n\) is similar, as \(T_n = \inf_{K:n} s_K\) is non-decreasing.

Remark: \(\limsup s_n \leq \sup s_n\), \(\liminf s_n \geq \inf s_n\).

But need not be equal (as in Example 4 on p. 36).

\(\limsup s_n\) is the value that \(\lim\) many of \(s_n\) need to get close to.
\(\liminf s_n\) (or \(\lim\) supremum of \(s_n\)).

Theorem (14): Let \(S = \limsup s_n\), \(T = \liminf s_n\).

There exists a subsequence \(s_{n_k} \to S\) as \(k \to \infty\).

There exists a subsequence \(s_{j_l} \to T\) as \(j \to \infty\).

Check for the examples on p. 36.

Proof (for finite \(S, T\); exercises: prove for \(S, T = \pm \infty\) and for \(\liminf\)).

Proof for \(\sup\):

By def., \(S = \lim s_n\) where \(s_n = \sup_{K>n} s_K\).

Since \(S_0 = \sup_{K>0} s_K\), there exists \(n_1 \in N\) such that \(S_1 \leq S_{n_1} \leq S_0\).

Similarly, since \(S_1 = \sup_{K>n} s_K\), there exists \(n_2 \geq n_1\) such that \(S_2 - \frac{1}{2} \leq S_{n_2} \leq S_{n_1}\).

There exist \(n_k > n_{k-1}\) such that \(S_{n_k} - \frac{1}{2^{k-1}} \leq S_{n_k} \leq S_{n_{k-1}}\).

Take limit as \(k \to \infty\) and use Squeeze Thm. \(S_{n_k} \to S\) by def., hence \(s_{n_k} \to S\) as well.

QED.
Theorem 10.7. For a sequence \((s_n)\),

\[(S_n) \text{ converges and } \lim s_n = s\]

if and only if \(\lim \inf s_n = \lim \sup s_n = s\).

Proof (\(\Rightarrow\)). Suppose \(\lim s_n = s\), \(\lim \sup s_n = s'\). W.T.S.: \(s = s'\).

By Theorem 9.6 (E11.7, p.37), there exists a subsequence \(S_{n_k} \to s'\) as \(k \to \infty\).

But since \(\lim s_n = s\), \(S_{n_k} \to s\). Hence \(s = s'\).

The argument for \(\lim \inf s_n\) is similar. (Give it!)

(\(\Leftarrow\)). Assume \(\lim \inf s_n = \lim \sup s_n = s\). W.T.S.: \(s_n \to s\). We check that \((s_n)\) is Cauchy.

Let \(\varepsilon > 0\); by def. of liminf, limsup we can choose \(N\) such that

\[
\left| \sup_{n \geq N} s_n - s \right| < \frac{\varepsilon}{2}, \quad \left| \inf_{n \geq N} s_n - s \right| < \frac{\varepsilon}{2}
\]

for \(n, m > N\) (\(\star\)).

Let \(n, m > N\), with \(s_n > s_m\).

\[
\left| s_n - s_m \right| = s_n - s_m < s + \left( s + \frac{\varepsilon}{2} \right) - \left( s - \frac{\varepsilon}{2} \right) = \varepsilon.
\]

Thus \((s_n)\) is Cauchy, hence converges. Moreover, by Theorem 9.5 (E11.7, p.37), some subsequence of \((s_n)\) converges to \(s\). Hence \((s_n)\) converges to \(s\). Q.E.D.

Theorem 9.11.7. \(\lim \inf s_n\) is the largest value to which some subsequence of \((s_n)\) converges.

\(\lim \sup s_n\) is the smallest value to which some subsequence of \((s_n)\) converges.

Precisely, \(s_n \to s\) as \(k \to \infty\) then

\[
\lim \inf s_n \leq s \leq \lim \sup s_n.
\]

Proof. Let \(s_n \to s\) as \(k \to \infty\). Fix \(N\) and choose \(k\): \(n_k > N\). Then

\[
\inf_{j \geq N} s_j < s_{n_k} \leq \sup_{j \geq N} s_j.
\]

Take limits of both sides as \(k \to \infty\); by a comparison theorem,

\[
\inf_{j \geq N} s_j \leq s \leq \sup_{j \geq N} s_j.
\]

Now take limit of both sides as \(N \to \infty\); we obtain \(\lim \inf s_n \leq s \leq \lim \sup s_n\). Q.E.D.