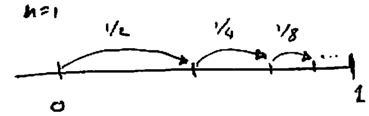
§14. Series.

Informally, series are sums of ∞ many terms. $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$



Examples: (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{n=1}^{\infty} 2^{-n} = 1.$

("A geometric series")

(b) $\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (will not compute this; see Complex Analysis)

(c) $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (as we shall see later).

Def (series) Let $(a_k)_{k=1}^{\infty}$ be a sequence of real numbers.

- The partial sums are defined as

$$s_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k, \qquad n \in \mathbb{N}.$$

- If $\lim s_n = S$ exists for some S (here $S \in \mathbb{R}$ or $S \in \{\pm\infty\}$), then we say that the series $\sum_{k=1}^{\infty} a_k$ converges, and write

$$\sum_{k=1}^{\infty} a_k = S.$$

(sometimes write $\sum_k a_k$ or $\sum a_k$).

- Otherwise, "diverges".

Remark: Similarly, for $\sum_{k=k_0}^{\infty} a_k$ (i.e. $s_n = a_{k_0+1} + a_{k_0+2} + \dots + a_n$.)

Examples (a) $\sum_{k=1}^{\infty} 2^{-k} = 1$

$\ulcorner$ $s_n = \sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}$ (by induction or geometric progression — CHECK!)

$\therefore \lim s_n = 1.$ Q.E.D. $\lrcorner$

(b) $\sum_{k=1}^{\infty} (-1)^k$ diverges

$\ulcorner$ $s_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$ $\Rightarrow$ (s_n) does not converge as $n \to \infty$. Q.E.D. $\lrcorner$

Theorem If $\sum_n a_n$ and $\sum_n b_n$ converge then

(a) $\sum_n (a_n + b_n) = \sum_n a_n + \sum_n b_n$

(b) $\sum_n k a_n = k \cdot \sum a_n, \quad k \in \mathbb{R}$

Exercise: Give the proof of Theorem using the limit theorems. (Ex. 14.5).

THM 14.5 If $\sum_{k=1}^{\infty} a_k$ converges then $\lim a_k = 0$.

Proof Assume $\sum_{k=1}^{\infty} a_k = S$.

By def, $s_n = \sum_{k=1}^{n} a_k \to S$ as $n \to \infty$

$\Rightarrow$ $S_n - S_{n-1} \to 0$ as $n \to \infty$ (as $\lim S_n = \lim S_{n-1}$).

But $S_n - S_{n-1} = a_n$. Q.E.D.

~~Examples $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$ diverges as lim~~

We can state Thm 14.5 as a

$\boxed{\text{DIVERGENCE TEST: If } a_n \not\to 0 \text{ as } n\to\infty \text{ then } \sum a_n \text{ diverges.}}$

Example (a) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$ diverges because $\left(1-\frac{1}{n}\right)^n \to e \neq 0$

(b) $\sum_{n=1}^{\infty} a^n$ diverges for $|a| \geq 1$ because $a_n \begin{cases} \to \infty, & |a|>1 \\ = 1 \neq 0, & a=1 \\ = (-1)^n \not\to 0, & a=-1 \end{cases}$

~~Example Geometric series~~

Proposition (Geometric series) [Example 1]

- If $|a|<1$ then the "geometric series" $\sum_{n=1}^{\infty} a^n$ converges and
$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$
- If $|a| \geq 1$ then the geometric series diverges.

Proof Let $|a|<1$. By Example 9.18 (geometric progression),

$$S_n = \sum_{k=1}^{n} a^k = 1+a+\dots+a^n = \frac{1-a^{n+1}}{1-a} \quad (\text{for } a \neq 1)$$

Then $$\lim S_n = \frac{1-\lim a^{n+1}}{1-a} \quad (\text{by the limit theorems})$$

$$= \frac{1-0}{1-a} = \frac{1}{1-a} \quad (\text{by } 9.7(B)).$$

- The case $|a| \geq 1$ was considered in Example (b) above.

Q.E.D

Example: $\sum_{n=1}^{\infty} 2^{-n} = \left(\sum_{n=0}^{\infty} 2^{-n}\right) - 1 = \frac{1}{1-1/2} - 1 = 1$.

Thm 14.3 (Cauchy Criterion) $\sum_{k=1}^{\infty} a_k$ converges if and only if:

for every $\varepsilon > 0$ there exists N such that

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \quad \text{for all} \quad n \geq m > N .$$

Proof The sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges if and only if (s_n) is Cauchy. The latter means that for every $\varepsilon > 0$ there exists N such that

$$\underbrace{|s_n - s_{m-1}|}_{= \sum_{k=m}^{n} a_k .} < \varepsilon \quad \text{for all} \quad n \geq m > N$$

Q.E.D.

[Compare to the def. of Cauchy! Why can we change m to $m-1$ and "$n, m > N$" to "$n \geq m > N$"?]

Thm 14.6 (Comparison Test). Let $a_n \geq 0$ for all n.

(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges,

(ii) If $\sum a_n$ diverges and $b_n \geq a_n$ for all n then $\sum b_n$ diverges.

Proof (i) For $n > m$,

$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \qquad \text{(by triangle inequality – Ex. 3.6(b))}$$

$$(*) \qquad \leq \sum_{k=m}^{n} a_k = \left| \sum_{k=m}^{n} a_k \right| \qquad (\text{since } a_k \geq 0).$$

Since $\sum a_n$ converges, it satisfies Cauchy criterion, hence $\sum b_n$ satisfies Cauchy criterion by $(*)$ hence $\sum b_n$ converges.

(ii) By contradiction; assume $\sum b_n$ converges; since $b_n \geq a_n \geq 0$, we apply (i) with the roles of a_i and b_i interchanged $\Rightarrow$ $\sum a_n$ converges. Contradiction.

Q.E.D.

Examples (a). $\sum_n \frac{n}{2^n}$: $\frac{n}{2^n} \leq \frac{1.5^n}{2^n} = \left(\frac{3}{4}\right)^n$ for all large n (by Ex. 9.14). Since $\sum (3/4)^n$ converges, so does $\sum \frac{n}{2^n}$ by comparison test.

<u>Example (6)</u> $\sum \frac{1}{n^2}$. Converges?

$$\frac{1}{n^2} < \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}.$$ By Comparison Test, it suffices to show that $\sum_{k=2}^{\infty}\left(\frac{1}{k-1}-\frac{1}{k}\right)$ converges

$$\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right) = \left(1-\cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}}-\frac{1}{n}\right) = 1-\frac{1}{n}.$$

("Telescoping series")

Since $\lim\left(1-\frac{1}{n}\right) = 1 \Rightarrow$ converges. Q.E.D.

<u>Remark</u> : $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ ~~th~~ (can be proved by complex analysis)