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Series ctd. (§14)

Corollary (to the Comparison test).

If $\sum_n |a_n|$ converges then $\sum_n a_n$ converges.

Proof: Apply Comparison Test for $a_n = |b_n|$. QED.

~~Example~~

Remark A series $\sum_n a_n$ such that $\sum |a_n|$ converges is called <u>absolutely convergent</u>.

Corollary states that absolutely convergent series converge.

Example: $\sum_n \frac{(-1)^{3n+1}}{n^2}$ converges because it converges absolutely.

$\sum_n \frac{1}{n^2} < \infty$ by Example on p. 42.

Root and Ratio Tests (§14)

Thm 14.9 (Root Test) [Cauchy]: Let $r = \limsup |a_n|^{1/n}$. The series $\sum_n a_n$:

(i) converges absolutely if $r < 1$;

(ii) diverges if $r > 1$.

Remark: the test is inconclusive if $r = 1$.

~~Interpr~~ <u>Idea</u> of the test: assume $r = |a_n|^{1/n}$ for all n; then

$|a_n| = r^n \Rightarrow \sum |a_n| = \sum_n r^n$ geometric series $\Rightarrow \begin{cases} \text{converges if } r<1 \\ \text{diverges if } r>1. \end{cases}$

So the Root Test <u>compares $\sum a_n$ to a geometric series</u>.

Proof (i) Assume $r < 1$. Let $\varepsilon > 0$; there exists N_0 such that

$\left| \sup_{n > N} |a_n|^{1/n} - r \right| < \varepsilon$ for $N > N_0$ (by def of lim sup)

So $\sup_{n > N_0} |a_n|^{1/n} < r + \varepsilon \Rightarrow |a_n| < (r+\varepsilon)^n$ for $n > N_0$

Choose $\varepsilon > 0$ so that $r + \varepsilon < 1$; hence $\sum_n (r+\varepsilon)^n$ converges (geometric)

Hence $\sum |a_n| < \sum (r+\varepsilon)^n$ converges by Comparison test, (absolutely) (ii). Assume $r>1$. Then by Corollary 11.4 there exists a subsequence $|a_{n_k}|^{1/n_k}$ that converges to $r>1$ as $k\to\infty$

~~For~~ $\cancel{\left| |a_{n_k}|^{1/n_k} - r \right| < \varepsilon}$ ~~for $k>$~~

Hence there exist K $|a_{n_k}|^{1/n_k} > 1$ for all $k>K$ by Exercise 8.10.

So $|a_{n_k}|>1$ for $k>K$, hence $\lim \{a_n\} \neq 0$.

Theorem 14.5 yields that $\sum a_n$ diverges. QED)

~~Exercises~~

Examples (a) $\sum_n \frac{n}{2^n}$

Root Test: $\left(\frac{n}{2^n}\right)^{1/n} = \frac{n^{1/n}}{2} \to \frac{1}{2}$ (by Prop. 9.7(c)).

By Root Test, $\sum_n \frac{n}{2^n}$ converges (absolutely)

(b). $\sum_k \left(\frac{k}{k+1}\right)^{k^2}$.

Root test: $a_k^{1/k} = \left(\frac{k}{k+1}\right)^k = \cancel{\left(1-\frac{1}{k}\right)^k}$

$= \frac{1}{\left(\frac{k+1}{k}\right)^k} = \frac{1}{\left(1+\frac{1}{k}\right)^k} \to \frac{1}{e}$ (by Example p. 28) $e=2.71828\ldots$

Since $\frac{1}{e}<1$, by Root Test the series converges (absolutely.)

~~(d) $\sum_{k=0}^\infty \frac{1}{k!}$ converges by Comparison test as $\frac{1}{k!}<\frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ converges (Example p. 42).~~

~~In fact, $\sum_{k=0}^\infty \frac{1}{k!} = e$.~~

(c) $\sum_{n=1}^\infty \frac{1}{n}$ -? $\left(\frac{1}{n}\right)^{1/n} \to 1$ (by Prop 9.7(c)). Root test is inconclusive.

$\sum \frac{1}{n^2}$ ———— same (inconclusive).

(d). $\sum_{n=1}^\infty (2+(-1)^n)^n x^n$. For which $x\in\mathbb{R}$ does the series converge?

$r = \limsup |(2+(-1)^n)^n x^n|^{1/n} = |x| \cdot \limsup (2+(-1)^n) = 3|x|$.

So, converges for $|x|<1/3$; diverges for $|x|>1/3$.

For $|x|=\frac{1}{3}$ we have for even $n=2k$: $(2+(-1)^{2n})^{2n} \cdot \left(\frac{1}{3}\right)^{2n} = 3^{2n} \cdot \left(\frac{1}{3}\right)^{2n} = 1$.

 Diverges for $|x|=1/3$ (by divergence test).

Thm (§14.9). (Ratio Test) [D'Alembert].

> Assume $r = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists. Then the series $\sum a_n$:
>
> (i) converges absolutely if $r<1$
>
> (ii) diverges if $r>1$.

$\rightarrow$ Recall a similar "Ratio test" for convergence of sequences, Ex. 9.12 (p. 26). It stated that $a_n \to 0$ in (i) ~~$|a_n| \to \infty$ in (ii)~~ $\Rightarrow$ Theorem is stronger.

Remark. The test is inconclusive for $r=1$.

Proof (i) Let $r<1$. Then for every $\varepsilon>0$ $\exists N$ s.t.

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - r\right| < \varepsilon \quad \text{for } n>N.$$

$$\Rightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| < r+\varepsilon \quad \text{for } n>N.$$

Choose $\varepsilon>0$ such that $r+\varepsilon<1$. Then

$$|a_{n+1}| \le (r+\varepsilon)|a_n|.$$

By induction, $|a_n| \le (r+\varepsilon)^{n-1}|a_1|$.

Since $\sum (r+\varepsilon)^{n-1}|a_1| = |a_1|\sum(r+\varepsilon)^{n-1}$ converges (geometric series, $\circ<r+\varepsilon<1$), the series $\sum |a_n|$ converges, too (by the Comparison Test).

(ii). Let $r>1$. Then there exist N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad \text{for } n>N \quad \text{(by Exercise 8.10)}.$$

Hence $|a_{n+1}| > |a_n|$, and by induction $|a_n| \ge |a_1|$.

Therefore $\lim a_n \neq 0$, and $\sum a_n$ diverges (by Divergence Test). QED

Examples (a) $\sum_k \frac{2^k}{k!}$.

Ratio test: $\left|\frac{2^{k+1}}{(k+1)!}\cdot\frac{k!}{2^k}\right| = \frac{2}{k} \to 0$ as $k\to\infty$. Converges absolutely

(b) $\sum_k \frac{x^k}{k!}$ converges absolutely for all $x\in\mathbb{R}$ (same argument).

In fact $\sum_k \frac{x^k}{k!} = e^x$ (later, from Taylor's formula).

(c) $\sum_n \frac{n!}{n^n}$. Ratio test: $\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right| = \left|\frac{(n+1)n^n}{(n+1)^{n+1}}\right| = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ (see Ex. (b) on p. 44).

Since $\frac{1}{e}<1$, the series converges absolutely

(d) Define $a_1=1$, $a_{n+1} = (-1)^n(1+\sqrt{n})^{-1}a_n$, $n\ge 1$. $\sum_n a_n$?

By Ratio Test: $|(1+\sqrt{n})^{-1}| \to 0 \Rightarrow$ the series converges absolutely.